

SPACES WITH \mathbb{G}_m -ACTION, HYPERBOLIC LOCALIZATION AND NEARBY CYCLES

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ABSTRACT. We study families of algebraic spaces with \mathbb{G}_m -action and prove Braden's theorem [Br03], [DG15] on hyperbolic localization for arbitrary base schemes. As an application, we obtain that hyperbolic localization commutes with nearby cycles.

0. INTRODUCTION

Algebraic varieties X with an action of the multiplicative group \mathbb{G}_m are a classical object of study [BB73]. The \mathbb{G}_m -action induces two stratifications on X : the strata of points X^+ floating to the fixed points, and the strata of points X^- floating away from the fixed points. Comparing these stratifications implies strong symmetry properties on the cohomology of these varieties, e.g. if X/\mathbb{F}_p is a proper variety with isolated \mathbb{G}_m -fixed points, then the étale cohomology groups $H_{\text{ét}}^i(X_{\mathbb{F}_p}, \mathbb{Q}_\ell)$ for $\ell \neq p$ are pure. In [Br03] Braden proves a general theorem on localizing equivariant objects on X to the subspace of fixed points X^0 . The result is used in a number of places [MV07], [Ach11], [ACHR15], and has proven to be of importance for geometric methods in representation theory, e.g. induction and restriction of character sheaves. Braden's theorem is generalized by Drinfeld and Gaitsgory [DG15] to algebraic spaces over fields. In the present manuscript, we consider families of spaces with \mathbb{G}_m -action and study the behavior under base change. The main motivation is the commutation of hyperbolic localization with nearby cycles which is inspired by a result of Arkhipov-Bezrukavnikov [AB09, Thm. 4], and which is used in subsequent work to study the local geometry of moduli spaces of shtukas at places of bad reduction.

0.1. Statement of results. Let S be a scheme, and let X/S be an algebraic space in the sense of [StaPro]. If \mathbb{G}_m acts on X/S (trivial on S), there are the following three functors on the category of S -schemes

$$(0.1) \quad \begin{aligned} X^0: T &\longmapsto \operatorname{Hom}_{S^m}^{\mathbb{G}_m}(T, X) \\ X^+: T &\longmapsto \operatorname{Hom}_{S^m}^{\mathbb{G}_m}((\mathbb{A}_T^1)^+, X) \\ X^-: T &\longmapsto \operatorname{Hom}_{S^m}^{\mathbb{G}_m}((\mathbb{A}_T^1)^-, X), \end{aligned}$$

where $(\mathbb{A}_T^1)^+$ (resp. $(\mathbb{A}_T^1)^-$) is \mathbb{A}_T^1 with the usual (resp. opposite) \mathbb{G}_m -action. The functor X^0 is the functor of \mathbb{G}_m -fixed points in X , and X^+ (resp. X^-) is called the attractor (resp. repeller). Informally speaking X^+ (resp. X^-) is the space of points x such that the limit $\lim_{\lambda \rightarrow 0} \lambda \cdot x$ (resp. $\lim_{\lambda \rightarrow \infty} \lambda \cdot x$) exists. Note that the formation of X^0 and X^\pm commutes with arbitrary base change $S' \rightarrow S$.

In many cases, the \mathbb{G}_m -action on a space is locally linear, and we consider the following notion. We say that a \mathbb{G}_m -action on X/S is *étale locally linearizable* if there exists a \mathbb{G}_m -equivariant étale covering family $\{U_i \rightarrow X\}_i$, where the U_i are S -affine schemes with \mathbb{G}_m -action. By upcoming results of Alper-Hall-Rydh [AHR16], cf. §0.2 for more details, every \mathbb{G}_m -action on a

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quasi-separated algebraic space X/S locally of finite presentation is étale locally linearizable (no condition on S).

Theorem A. *Let S be a scheme, and let X/S be an algebraic space with an étale locally linearizable \mathbb{G}_m -action.*

- i) The functor X^0 is representable by a closed subspace of X .*
- ii) The functor X^\pm is representable by a X^0 -affine algebraic space.*
- iii) If X/S is locally of finite presentation (resp. quasi-compact; resp. quasi-separated; resp. separated; resp. is a scheme), so are X^0 and X^\pm .*

Let X/S be an algebraic space locally of finite presentation with an étale locally linearizable \mathbb{G}_m -action. There are maps locally of finite presentation of S -spaces

$$(0.2) \quad \begin{array}{ccc} & X^\pm & \\ q^\pm \swarrow & & \searrow p^\pm \\ X^0 & & X, \end{array}$$

where q^\pm is given by evaluating a morphism at the zero section, and p^\pm by evaluating a morphism at the unit section. Let $n > 1$ be a positive integer invertible on S , and denote by $D(X, \mathbb{Z}/n)$ the unbounded derived category of $(X_{\text{ét}}, \mathbb{Z}/n)$ -modules, where $X_{\text{ét}}$ is the étale topos of X . Let us define two functors from $D(X, \mathbb{Z}/n)$ to $D(X^0, \mathbb{Z}/n)$ by pull-push as follows

$$(0.3) \quad \begin{aligned} L_{X/S}^+ &\stackrel{\text{def}}{=} (q^+)_{!} \circ (p^+)^* \\ L_{X/S}^- &\stackrel{\text{def}}{=} (q^-)_* \circ (p^-)^!. \end{aligned}$$

As in Braden's work [Br03] (or Drinfeld-Gaitsgory's work [DG15]) there exists a natural transformation of functors

$$(0.4) \quad L_{X/S}^- \longrightarrow L_{X/S}^+.$$

Let $a, p: \mathbb{G}_{m,S} \times_S X \rightarrow X$ denote the action (resp. projection). We say a complex in $D(X, \mathbb{Z}/n)$ is (naively) \mathbb{G}_m -equivariant if there exists an isomorphism $a^* \mathcal{A} \simeq p^* \mathcal{A}$ in $D(\mathbb{G}_{m,S} \times_S X, \mathbb{Z}/n)$. Let us define $D(X, \mathbb{Z}/n)^{\mathbb{G}_m\text{-mon}}$ to be the full subcategory strongly generated by \mathbb{G}_m -equivariant complexes, i.e. generated by a finite iteration of taking the cone of a morphism.

Theorem B. *Let S be a scheme, and let X/S be an algebraic space locally of finite presentation with an étale locally linearizable \mathbb{G}_m -action. Let $\mathcal{A} \in D(X, \mathbb{Z}/n)^{\mathbb{G}_m\text{-mon}}$ be a bounded below complex.*

- i) The arrow of $D(X^0, \mathbb{Z}/n)$*

$$L_{X/S}^- \mathcal{A} \xrightarrow{\sim} L_{X/S}^+ \mathcal{A}$$

is an isomorphism. In particular, the complex $L_{X/S}^- \mathcal{A}$ is bounded below.

- ii) For any morphism of schemes $f: S' \rightarrow S$, the isomorphism in i) is compatible with base change along f_* and f^* . If f is locally of finite type, it is also compatible with $f_!$ and $f^!$.*

Let us point out the following consequence of Theorem B. Let S be the spectrum of a henselian discrete valuation ring with generic point η and special point s . Fix a geometric point $\bar{\eta} \rightarrow \eta$. Then there is the functor of nearby cycles

$$(0.5) \quad \begin{aligned} \Psi_X: D(X_\eta, \mathbb{Z}/n) &\longrightarrow D(X_s \times_S \eta, \mathbb{Z}/n) \\ \mathcal{A} &\longmapsto \bar{i}^* \bar{j}_* \mathcal{A}_{\bar{\eta}}, \end{aligned}$$

where $D(X_s \times_S \eta, \mathbb{Z}/n)$ is as in [SGA 7, XIII] the derived category of $((X_{\bar{s}})_{\text{ét}}, \mathbb{Z}/n)$ -modules with a continuous action of the Galois group compatible with its action on $X_{\bar{s}}$. The usual functorialities of nearby cycles give transformations from $D(X_{\eta}, \mathbb{Z}/n)$ to $D(X_s^0 \times_S \eta, \mathbb{Z}/n)$ as follows

$$(0.6) \quad \begin{aligned} L_{X_{\bar{s}}/\bar{s}}^- \circ \Psi_X &\longleftarrow \Psi_{X^0} \circ L_{X_{\eta}/\eta}^-, \\ L_{X_{\bar{s}}/\bar{s}}^+ \circ \Psi_X &\longrightarrow \Psi_{X^0} \circ L_{X_{\eta}/\eta}^+. \end{aligned}$$

Corollary. *Let S be the spectrum of a henselian discrete valuation ring, and let X/S be an algebraic space of finite type with an étale locally linearizable \mathbb{G}_m -action. Then, for $\mathcal{A} \in D(X_{\eta}, \mathbb{Z}/n)$ bounded below, there is a commutative diagram in $D(X_s^0 \times_S \eta, \mathbb{Z}/n)$*

$$(0.7) \quad \begin{array}{ccc} L_{X_{\bar{s}}/\bar{s}}^- \circ \Psi_X(\mathcal{A}) & \longleftarrow & \Psi_{X^0} \circ L_{X_{\eta}/\eta}^-(\mathcal{A}) \\ \downarrow & & \downarrow \\ L_{X_{\bar{s}}/\bar{s}}^+ \circ \Psi_X(\mathcal{A}) & \longrightarrow & \Psi_{X^0} \circ L_{X_{\eta}/\eta}^+(\mathcal{A}), \end{array}$$

and all arrows are isomorphisms if \mathcal{A} is \mathbb{G}_m -monodromic.

0.2. Link to the literature and strategy of proof. The commutativity of hyperbolic localization with nearby cycles is a purely formal consequence of Theorem B. Hence, sufficient generality is of importance: there are no finiteness assumptions imposed neither on the base scheme S (e.g. locally noetherian) nor on the sheaves (e.g. constructible). Let us link the results to the literature¹.

Theorem A. The spaces X^{\pm} in (0.1) are defined by Drinfeld [Dr13], and he proves representability of X^{\pm} for quasi-separated algebraic spaces of finite type over fields. Note that similar functors are studied by Hesselink [He80], where he proves representability under the existence of a \mathbb{G}_m -invariant affine open cover, i.e. the \mathbb{G}_m -action is Zariski locally linearizable. By results of Sumihiro [Sum74, Cor. 2], [Sum75, Cor. 3.11] every \mathbb{G}_m -action on a normal variety has this property. Note that this fails without normality assumption, e.g. for \mathbb{P}^1 with 0 and ∞ identified. More recently, Alper-Hall-Rydh [AHR15, Thm. 2.24] recover Drinfeld's result on X^{\pm} by proving general results on the representability of Hom-stacks. The condition of being étale locally linearizable comes from their generalization of Sumihiro's result [AHR15, Thm. 2.5]: every \mathbb{G}_m -action on a quasi-separated algebraic space locally of finite presentation is étale locally linearizable. In [loc. cit.] this is shown for algebraically closed fields as bases and in upcoming work of Alper-Hall-Rydh [AHR16] this hypothesis is removed. Theorem A is straight forward:

- (1) Prove Theorem A for S -affine schemes.
- (2) Descend the representability and favorable properties by using an equivariant atlas $\{U_i \rightarrow X\}$.

Note that if X is a scheme where the \mathbb{G}_m -action is not Zariski locally linearizable, then the argument for Theorem A does not simplify, i.e. that X^{\pm} is a scheme follows a posteriori from X^{\pm} being an algebraic space. Note that our method is very close to Alper-Hall-Rydh's arguments [AHR15, §5.12]. We choose to include Theorem A because it makes the present manuscript self contained, and because we think it is of interest in its own: the hypothesis of being étale locally linearizable can be verified by hand in many cases.

Theorem B. Braden [Br03] proves that for a normal variety X over an algebraically closed field, the transformation (0.4) is an isomorphism on weakly \mathbb{G}_m -equivariant complexes. Using Sumihiro's theorem, he reduces to the case of an affine space with a linear \mathbb{G}_m -action, and then

¹The author includes what he knows, but the outline is probably not complete. The author is grateful for every comment, e.g. if the reader feels that formulations are incorrect or other work should be mentioned. Of course, other comments or questions are equally welcome.

uses a contraction argument [Br03, Lem. 6]. In [DG15], Drinfeld-Gaitsgory extend Braden's result to the case of quasi-separated algebraic spaces locally of finite type over characteristic zero fields in the context of D -modules. Their argument uses a certain family $\tilde{X} \rightarrow \mathbb{A}^1$, which is shown to be representable in [Dr13], and a sufficiently good six functor formalism, e.g. existence of a dualizing complex. Their method applies to the étale topology using \mathbb{Q}_ℓ -sheaves with constructible cohomologies over fields of characteristic $\neq \ell$ [DG15, §0.4]. In our approach, we follow Braden's original method:

- (1) Prove that Theorem B i) holds for affine spaces with a linear \mathbb{G}_m -action.
- (2) Reduce to case (1) using an equivariant atlas $\{U_i \rightarrow X\}_i$: pull back to U_i and embed U_i into an affine space with a linear \mathbb{G}_m -action.

A careful analysis of Braden's argument shows that in the presence of torsion coefficients no finiteness assumptions neither on S nor on the sheaves are necessary. Theorem B ii) is proven by a diagram chase. The isomorphisms in Theorem B ii) are due to strong symmetry properties implied by Theorem B i), e.g. f_* commutes with L^- and, by three out of four, it also commutes with L^+ on monodromic complexes.

0.3. Structure of the manuscript. In §1, we study spaces with an étale locally linearizable \mathbb{G}_m -action, and prove Theorem A, cf. Theorem 1.8 below. Paragraph §2 is devoted to the proof of Theorem B i), cf. Theorem 2.6 below. The toy case is \mathbb{A}_S^1 , cf. §2.4, and the argument for affine spaces with a linear \mathbb{G}_m -action in §2.5 and §2.6 builds upon it. In §2.7 and §2.8, we deduce Theorem B i) from the latter case using a \mathbb{G}_m -equivariant atlas. The functorial properties in Theorem B ii) are studied in §3.1, and the commutation of hyperbolic localization with nearby cycles is deduced in §3.2.

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0.5. Notation. For a scheme S , we denote by (Sch/S) the category of S -schemes. By a space X/S , we mean an algebraic space X/S in the sense of [StaPro, Tag 025Y]: a sheaf on the big fppf-site

$$X: (Sch/S)_{\text{fppf}}^{\text{op}} \longrightarrow \text{Set}$$

with representable diagonal and which admits a surjective étale map from a scheme. In particular, we do not assume X to be quasi-separated. Throughout we fix a general base scheme S . Special hypothesis on S are spelled out explicitly when needed. For two sheaves X and Y on $(Sch/S)_{\text{fppf}}$, we denote by $X \times Y = X \times_S Y$ the fiber product and by $\text{Hom}_S(Y, X)$ the set of S -morphisms.

1. SPACES WITH ÉTALE LOCALLY LINEARIZABLE \mathbb{G}_m -ACTION

1.1. General nonsense. Let S be a scheme, and let X/S be a space. For a S -scheme T , let $X_T = X \times T$. Define the contravariant set-valued functor $\underline{\text{Hom}}_S(Y, X)$ on $(Sch/S)_{\text{fppf}}$, for any S -scheme T , by

$$\underline{\text{Hom}}_S(Y, X): T \longmapsto \text{Hom}_T(Y_T, X_T),$$

Note that $\text{Hom}_T(Y_T, X_T) = \text{Hom}_S(Y_T, X)$. The functor $\underline{\text{Hom}}_S(Y, X)$ is a sheaf on $(Sch/S)_{\text{fppf}}$. For a morphism $f: X' \rightarrow X$ of S -spaces, there is a transformation as follows

$$(1.1) \quad \underline{\text{Hom}}_S(Y, X') \longrightarrow \underline{\text{Hom}}_S(Y, X), \quad x \longmapsto f \circ x.$$

For a morphism $g: Y' \rightarrow Y$ of S -spaces, there is a transformation as follows

$$(1.2) \quad \underline{\text{Hom}}_S(Y, X) \longrightarrow \underline{\text{Hom}}_S(Y', X), \quad x \longmapsto x \circ g.$$

For a morphism $S' \rightarrow S$ of schemes, there is an isomorphism as follows

$$(1.3) \quad \underline{\mathrm{Hom}}_{S'}(Y_{S'}, X_{S'}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_S(Y, X) \times S',$$

which is compatible with (1.1) and (1.2).

Let G/S be a fppf-sheaf of groups. If X/S and Y/S are equipped with a (left) G -action, then G acts on $\underline{\mathrm{Hom}}_S(Y, X)$: for any S -scheme T and $(g, x) \in G(T) \times \underline{\mathrm{Hom}}_S(Y, X)(T)$ define $g * x$ by

$$g * x \stackrel{\mathrm{def}}{=} g \circ x \circ g^{-1},$$

where g (resp. g^{-1}) denotes the automorphism $X_T \rightarrow X_T$ (resp. $Y_T \rightarrow Y_T$) given by the G -action on X (resp. Y). Define the subfunctor $\underline{\mathrm{Hom}}_S^G(Y, X)$ of G -equivariant morphisms from Y to X , for any S -scheme T by

$$\underline{\mathrm{Hom}}_S^G(Y, X): T \mapsto \{x \in \mathrm{Hom}_T(Y_T, X_T) \mid \forall g \in G(T) : g * x = x\}.$$

In other words, $\underline{\mathrm{Hom}}_S^G(Y, X)$ is the subfunctor of G -fixed points in $\underline{\mathrm{Hom}}_S(Y, X)$.

Lemma 1.1. *Let G/S be a fppf-sheaf of groups. Let X/S and Y/S be spaces with G -action.*

- i) *The functor $\underline{\mathrm{Hom}}_S^G(Y, X)$ is a subsheaf of $\underline{\mathrm{Hom}}_S(Y, X)$ on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.*
- ii) *For a G -equivariant morphism $X' \rightarrow X$ (resp. $Y' \rightarrow Y$) of S -spaces, the transformation (1.1) (resp. (1.2)) restricts to a morphism on subsheaves*

$$\underline{\mathrm{Hom}}_S^G(Y, X') \rightarrow \underline{\mathrm{Hom}}_S^G(Y, X) \quad (\text{resp. } \underline{\mathrm{Hom}}_S^G(Y, X) \rightarrow \underline{\mathrm{Hom}}_S^G(Y', X)).$$

- iii) *For a morphism $S' \rightarrow S$ of schemes, the isomorphism (1.3) restricts to an isomorphism of subsheaves*

$$\underline{\mathrm{Hom}}_{S'}^G(Y_{S'}, X_{S'}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_S^G(Y, X) \times S',$$

which is compatible with the transformations constructed in ii).

Proof. For i), note that $\underline{\mathrm{Hom}}_S^G(Y, X)$ is the functor of fixed points with respect to the G -action on $\underline{\mathrm{Hom}}_S(Y, X)$, and hence a sheaf. Parts ii) and iii) are immediate. \square

Example 1.2. i) If $Y = S$, then $\underline{\mathrm{Hom}}_S^G(Y, X) = X^G$ is the functor of fixed points. If X/S is a quasi-compact quasi-separated scheme and G/S is a flat group scheme, then X^G is representable by a closed subscheme of X , cf. Fogarty [F73].

ii) If $Y = G$ with the translation action, then $\underline{\mathrm{Hom}}_S^G(Y, X) = X$ by evaluating a morphism at the unit section of G .

1.2. Attractors, repellers and fixed points. We are interested in the case where $G = \mathbb{G}_m$ is the multiplicative group viewed as a sheaf of groups on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. Consider the following examples of schemes with \mathbb{G}_m -action:

- i) $Y = S$ equipped with the trivial \mathbb{G}_m -action;
- ii) $Y = (\mathbb{A}_S^1)^+$ where the underlying scheme is \mathbb{A}_S^1 equipped with the \mathbb{G}_m -action by dilations, i.e. for any S -scheme T and $\lambda \in \mathbb{G}_{m,S}(T) = \mathcal{O}_T^\times$, $x \in \mathbb{A}_S^1(T) = \mathcal{O}_T$ the action is given by $(\lambda, x) \mapsto \lambda \cdot x$;
- iii) $Y = (\mathbb{A}_S^1)^-$ where the underlying scheme is \mathbb{A}_S^1 equipped with the opposite \mathbb{G}_m -action, i.e. the action is given by $(\lambda, x) \mapsto \lambda^{-1} \cdot x$.

Drinfeld [Dr13] introduces the following notations.

Definition 1.3. Let X/S be a space with \mathbb{G}_m -action. Define the sheafs X^0 , X^+ and X^- on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ by

$$\begin{aligned} X^0 &\stackrel{\mathrm{def}}{=} \underline{\mathrm{Hom}}_S^{\mathbb{G}_m}(S, X); \\ X^+ &\stackrel{\mathrm{def}}{=} \underline{\mathrm{Hom}}_S^{\mathbb{G}_m}((\mathbb{A}_S^1)^+, X); \\ X^- &\stackrel{\mathrm{def}}{=} \underline{\mathrm{Hom}}_S^{\mathbb{G}_m}((\mathbb{A}_S^1)^-, X). \end{aligned}$$

The sheaf X^0 is called the *space of fixed points*, X^+ the *attractor* and X^- the *repeller*.

Remark 1.4. i) The sheaf $X^0 = X^{\mathbb{G}_m}$ is the functor of fixed points as in Remark 1.2, i) above. In case ii) (resp. iii)), the sheaf X^+ (resp. X^-) is the functor of points *floating to* (resp. *away from*) the fixed points. Informally speaking, the limit

$$\lim_{\lambda \rightarrow 0} \lambda \cdot x \quad (\text{resp. } \lim_{\lambda \rightarrow \infty} \lambda \cdot x)$$

should exist.

ii) Note that X^0 and X^\pm inherit \mathbb{G}_m -actions from X (the trivial one on X^0), and with respect to these actions $(X^\pm)^0 = X^0$.

Example 1.5. i) Let $X = \mathbb{P}_S^1$ with the natural \mathbb{G}_m -action. Then $X^0 = \{0_S\} \amalg \{\infty_S\}$ are the fixed points, $X^+ = (\mathbb{A}_S^1)^+ \amalg \{\infty_S\}$ and $X^- = (\mathbb{A}_S^1)^- \amalg \{0_S\}$. In particular, X^+ and X^- are representable.

ii) Let $X = G$ be a S -group scheme with \mathbb{G}_m -action given by conjugation with a cocharacter $\lambda: \mathbb{G}_{m,S} \rightarrow G$. Then $X^0 = Z_G(\lambda)$ is the centralizer of λ and $X^\pm = P(\pm\lambda)$ are the ‘*parabolic*’ subgroups defined by the dynamic method, cf. [Co14, Thm. 4.1.7].

Definition 1.6. Let S be a scheme, and let X/S be a space. A \mathbb{G}_m -action on X is called *étale locally linearizable* if there exists a \mathbb{G}_m -equivariant covering family

$$(1.4) \quad \{U_i \longrightarrow X\}_i,$$

where U_i are S -affine schemes with \mathbb{G}_m -action and the maps $U_i \rightarrow X$ are étale.

Remark 1.7. i) That the family $\{U_i \rightarrow X\}_i$ is covering means that the map $\coprod_i U_i \rightarrow X$ is surjective on the underlying topological spaces.

ii) The attribute ‘*linearizable*’ refers to the fact that an affine scheme of finite presentation with \mathbb{G}_m -action can be (Zariski locally on the base) equivariantly embedded as a closed subscheme into some affine space on which \mathbb{G}_m -acts linearly, cf. Lemma 2.21 below.

iii) If S is the spectrum of an algebraically closed field, then every \mathbb{G}_m -action on a quasi-separated algebraic space X/S locally of finite presentation is étale locally linearizable, cf. [AHR15, Thm. 2.5]. In forthcoming work of Alper-Hall-Rydh [AHR16] étale locally linearizability is shown for an arbitrary base scheme S .

Theorem 1.8. *Let S be a scheme, and let X/S be a space with an étale locally linearizable \mathbb{G}_m -action. Let $\{U_i \rightarrow X\}_i$ be a S -affine \mathbb{G}_m -equivariant étale covering family.*

i) *The subfunctor X^0 of X is representable by a closed subspace, and the induced family $\{U_i^0 \rightarrow X^0\}_i$ is S -affine, étale and covering.*

ii) *The functors X^\pm are representable by algebraic spaces, and the induced family $\{U_i^\pm \rightarrow X^\pm\}_i$ is S -affine, étale, \mathbb{G}_m -equivariant and covering.*

iii) *If X/S is locally of finite presentation (resp. quasi-compact; resp. quasi-separated; resp. separated; resp. is a scheme), so are X^0 and X^\pm .*

This theorem, combined with Corollary 1.12 below, implies Theorem A from the introduction. The proof of part i) is in §1.4 and of part ii) and iii) in §1.5 below. The strategy is to descend the desired properties from the equivariant atlas. The keys are Lemmas 1.10 and 1.11, cf. also [AHR15, Lem. 5.8] over fields. As it turns out X^\pm is an affine X^0 -space (Corollary 1.12) which implies part iii). Let us warm up with an easy case.

1.3. The affine case. Let $X = \text{Spec}_S(\mathcal{B})$ be an S -affine scheme where \mathcal{B} denotes a quasi-coherent \mathcal{O}_S -algebra. If S is connected, a \mathbb{G}_m -action on X/S is equivalent to a \mathbb{Z} -grading

$$(1.5) \quad \mathcal{B} = \bigoplus_{i \in \mathbb{Z}} \mathcal{B}_i$$

on the \mathcal{O}_S -algebra \mathcal{B} , i.e. (1.5) as quasi-coherent \mathcal{O}_S -modules and $\mathcal{B}_i \cdot \mathcal{B}_j \subset \mathcal{B}_{i+j}$. Let \mathcal{I}^+ (resp. \mathcal{I}^- , resp. \mathcal{I}^0) be the quasi-coherent² ideal sheaf in \mathcal{B} generated by the homogeneous elements of strictly negative (resp. strictly positive, resp. non-zero) degree.

Lemma 1.9. *i) The functor X^0 is representable by the closed subscheme of X defined by \mathcal{I}^0 .
ii) The functor X^\pm is representable by the closed subscheme of X defined by \mathcal{I}^\pm .*

Proof. Let $p: T \rightarrow S$ be a scheme. Since X is S -affine the set $X(T)$ identifies with set of \mathcal{O}_S -algebra morphisms $\mathcal{B} \rightarrow p_*\mathcal{O}_T$, and $X^0(T)$ is the subset of \mathbb{Z} -graded \mathcal{O}_S -algebra morphisms $\mathcal{B} \rightarrow p_*\mathcal{O}_T$, where $p_*\mathcal{O}_T$ is in degree 0. This implies part i). Likewise, the set $X^+(T)$ (resp. $X^-(T)$) identifies with the set of \mathbb{Z} -graded \mathcal{O}_S -algebra morphisms $\mathcal{B} \rightarrow p_*\mathcal{O}_T[t]$ where the parameter t has degree 1 (resp. -1). This implies part ii). \square

1.4. The space of fixed points X^0 . It is suprising that $X^0 \subset X$ is closed, even if X is not separated. This is closely related to the connectedness of \mathbb{G}_m . Let us prepare for the proof.

Lemma 1.10. *Let $U \rightarrow X$ be \mathbb{G}_m -equivariant étale S -morphism. Then as functors*

$$U^0 = U \times_X X^0.$$

Proof. Let T/S be a scheme. An element $\varphi \in (U \times_X X^0)(T)$ corresponds to a commutative diagram

$$\begin{array}{ccc} & U & \\ \tilde{f} \nearrow & \downarrow \text{ét} & \\ T & \xrightarrow{f} & X \end{array}$$

where f is \mathbb{G}_m -equivariant. We have to show that \tilde{f} is \mathbb{G}_m -equivariant. It is enough to show equivariance étale locally. If T is the spectrum of a strictly henselian local ring, then \tilde{f} is \mathbb{G}_m -equivariant (because \mathbb{G}_m is connected and $U \rightarrow X$ is étale). Since $U \rightarrow X$ is locally of finite presentation, we get

$$\mathrm{Hom}_X^{\mathbb{G}_m}(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Hom}_X^{\mathbb{G}_m}(T_i, U)$$

for any cofiltered limit $\lim_i T_i$ of affine schemes in $(Sch/X)_{\mathrm{fppf}}$. Hence, \tilde{f} is étale locally \mathbb{G}_m -equivariant. The lemma follows. \square

Proof of Theorem 1.8 i). Let us show that X^0 is representable by a closed subspace, and $\{U_i^0 \rightarrow X^0\}_i$ is S -affine, étale and covering. By Lemma 1.9 i), the U_i^0 are closed subschemes of U_i , hence S -affine. Lemma 1.10 shows that the following commutative diagram

$$(1.6) \quad \begin{array}{ccc} \coprod_i U_i^0 & \longrightarrow & X^0 \\ \downarrow & & \downarrow \\ \coprod_i U_i & \longrightarrow & X \end{array}$$

is cartesian. Now the representability of X^0 follows from [StaPro, Tag 03I2] applied to the transformation $X^0 \rightarrow X$ and the property ‘closed immersion’: (i) being a closed immersion is stable under base change, fppf-local local on the base and closed immersions satisfy fppf-descent (because they are affine); (ii) X^0 is a sheaf; (iii) X is an algebraic space; (iv) the bottom arrow in (1.6) is surjective and étale, and $\coprod_i U_i^0$ is representable; (v) the left vertical arrow in (1.6) is a closed immersion. This implies that X^0 is an algebraic space and $X^0 \rightarrow X$ is a closed immersion. This proves Theorem 1.8 i). \square

²The sheaf \mathcal{I}^+ is quasi-coherent because its the image of a quasi-coherent sheaf: $\mathcal{B} \times (\oplus_{i < 0} \mathcal{B}_i) \rightarrow \mathcal{B}$, $(b, c) \mapsto b \cdot c$.

1.5. Attractors and repellers X^\pm . For an étale locally linearizable \mathbb{G}_m -action the representability of X^\pm is proven similarly. Let us explain the argument.

Note that under the morphism $\mathbb{G}_m \rightarrow \mathbb{G}_m$, $\lambda \mapsto \lambda^{-1}$ the notions of X^+ and X^- are interchanged. Hence, it is enough to prove representability of X^+ . Let us denote $(\mathbb{A}_S^1)^+$ by \mathbb{A}_S^1 in this subsection.

The zero section $S \rightarrow \mathbb{A}_S^1$ is \mathbb{G}_m -equivariant and defines by functoriality a morphism $X^+ \rightarrow X^0$.

Lemma 1.11. *Let $U \rightarrow X$ be a \mathbb{G}_m -equivariant étale S -morphism where U is an S -affine scheme. Then as functors*

$$U^+ = U^0 \times_{X^0} X^+.$$

Proof. Let $p: T \rightarrow S$ be a scheme, and let $\varphi \in (U^0 \times_{X^0} X^+)(T)$. The element φ corresponds to a commutative diagram of \mathbb{G}_m -equivariant morphisms

$$\begin{array}{ccc} T & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathbb{A}_T^1 & \longrightarrow & X \end{array}$$

where $T \rightarrow \mathbb{A}_T^1$ is the zero section. Let us construct a unique \mathbb{G}_m -equivariant lift $\mathbb{A}_T^1 \rightarrow U$. For $i \geq 0$ denote $\mathbb{A}_{T,i}^1 = \text{Spec}_T(\mathcal{O}_T[t]/t^{i+1})$ the i -th infinitesimal neighbourhood of the zero section $T \rightarrow \mathbb{A}_T^1$. Since $U \rightarrow X$ is étale, there is a unique \mathbb{G}_m -equivariant lift $\text{colim}_i \mathbb{A}_{T,i}^1 \rightarrow U$. We claim that

$$(1.7) \quad \text{Hom}_{\mathbb{G}_m}^{\mathbb{G}_m}(\text{colim}_i \mathbb{A}_{T,i}^1, U) = \text{Hom}_{\mathbb{G}_m}^{\mathbb{G}_m}(\mathbb{A}_T^1, U).$$

Let $U = \text{Spec}_S(\mathcal{R})$ for a quasi-coherent \mathcal{O}_S -algebra \mathcal{R} . Assume that S is connected, and let $\mathcal{R} = \bigoplus_{i \in \mathbb{Z}} \mathcal{R}_i$ be the grading corresponding to the \mathbb{G}_m -action. An element of the left hand side of (1.7) corresponds to a morphism of \mathcal{O}_S -algebras

$$(1.8) \quad \mathcal{R} \longrightarrow p_* \mathcal{O}_T[[t]]$$

compatible with \mathbb{Z} -gradings (the parameter t has degree 1). As $\mathcal{R} = \bigoplus_{i \in \mathbb{Z}} \mathcal{R}_i$ each morphism (1.8) factors through the subalgebra $\bigoplus_{i \geq 0} p_* \mathcal{O}_T \cdot t^i = p_* \mathcal{O}_T[t]$ of $p_* \mathcal{O}_T[[t]]$, i.e. defines a \mathbb{Z} -graded \mathcal{O}_S -algebra morphism $\mathcal{R} \rightarrow p_* \mathcal{O}_T[t]$. This proves (1.7) and implies the lemma. \square

Proof of Theorem 1.8 ii). We show that X^+ is an algebraic space, and that the family $\{U_i^+ \rightarrow X^+\}$ is S -affine, étale and covering. Note that \mathbb{G}_m -equivariance follows from functoriality. By Lemma 1.9, the sheaf U_i^+ is representable by a closed subscheme of U_i . Lemma 1.11 shows that the following commutative diagram of sheaves

$$(1.9) \quad \begin{array}{ccc} \coprod_i U_i^+ & \longrightarrow & X^+ \\ \downarrow & & \downarrow \\ \coprod_i U_i^0 & \longrightarrow & X^0 \end{array}$$

is cartesian where $X^+ \rightarrow X^0$ (resp. $U_i^+ \rightarrow U_i^0$) is induced by the zero section $S \rightarrow \mathbb{A}_S^1$. By Theorem 1.8 i), the family $\{U_i^0 \rightarrow X^0\}_i$ is étale and covering. Now the representability of X^+ follows from [StaPro, TAG 03I2] applied to the transformation $X^+ \rightarrow X^0$ and the property ‘*affine*’: (i) being affine is stable under base change, fppf-local on the base and affine morphisms satisfy fppf-descent; (ii) X^+ is a sheaf; (iii) X^0 is an algebraic space by Theorem 1.8 i); (iv) the bottom arrow in (1.9) is surjective and étale, and $\coprod_i U_i^+$ is representable; (v) the left vertical arrow in (1.9) is affine. This implies that X^+ is an algebraic space and $X^+ \rightarrow X^0$ is affine. This proves Theorem 1.8 ii). \square

Corollary 1.12. *The map $X^+ \rightarrow X^0$ induced by the zero section $S \rightarrow \mathbb{A}_S^1$ is affine, has geometrically connected fibers and induces a bijection on the sets of connected components $\pi_0(|X^+|) \simeq \pi_0(|X^0|)$ of the underlying topological spaces.*

Proof. Affineness of $X^+ \rightarrow X^0$ is proven above, and we show that the fibers are connected. Let K be a field, and let $x: \operatorname{Spec}(K) \rightarrow X^0$ be a point. Denote $X_x^+ = X^+ \times_{X^0, x} \operatorname{Spec}(K)$. We claim that its underlying topological space $|X_x^+|$ is connected, cf. [StaPro, Algebraic Spaces, §4] for the definition of underlying topological spaces. Let $y: \operatorname{Spec}(L) \rightarrow X_x^+$ be a point, and denote by x_L the composition $\operatorname{Spec}(L) \rightarrow \operatorname{Spec}(K) \xrightarrow{x} X^0$. Then x_L and x define the same point of $|X_x^+|$. The key observation is that the \mathbb{G}_m -action on X^+ extends to an action of the multiplicative monoid \mathbb{A}^1 in the obvious way. Hence, the \mathbb{A}^1 -orbit of y defines a map $h: \mathbb{A}_L^1 \rightarrow X_x^+$ with $h(1) = y$ and $h(0) = x_L$ because $(X_x^+)^0 = x$, cf. Lemma 1.9 i). Thus, the points defined by y and x lie in the connected set $|h|(|\mathbb{A}_L^1|)$. Since y was arbitrary, this shows that $|X_x^+|$ is connected. In particular, the continuous map $|X^+| \rightarrow |X^0|$ has connected fibers, and the assertion on connected components follows from the existence of a section $|X^0| \subset |X^+|$. This shows the corollary. \square

Proof of Theorem 1.8 iii). Let us check the list of properties. If X is locally of finite presentation, then it is immediate from the definition that X^0 and X^+ are locally of finite presentation, i.e. the functors commute with cofiltered limits of affine schemes in $(Sch/S)_{\text{fppf}}$. Let \mathcal{P} be one of the following properties: quasi-compact, quasi-separated, separated, being a scheme. If X has property \mathcal{P} , so has X^0 (because $X^0 \subset X$ is closed). Since $X^+ \rightarrow X^0$ is affine, in particular representable, quasi-compact and (quasi-)separated, and each property is stable under composition, it follows that X^+ has property \mathcal{P} , if X^0 has. This finishes the proof of Theorem 1.8 iii). \square

1.6. The hyperbolic localization diagram. Let us relate the spaces X^0 and X^\pm to each other. The structure morphism $(\mathbb{A}_S^1)^\pm \rightarrow S$ is \mathbb{G}_m -equivariant which defines by functoriality, cf. Lemma 1.1 ii), a transformation

$$i^\pm: X^0 \longrightarrow X^\pm.$$

Further, the zero section $S \rightarrow (\mathbb{A}_S^1)^\pm$ is \mathbb{G}_m -equivariant which defines, again by functoriality, a transformation³

$$q^\pm: X^\pm \longrightarrow X^0,$$

such that $p^\pm \circ i^\pm = \text{id}$. Likewise, the inclusion $\mathbb{G}_{m,S} \rightarrow (\mathbb{A}_S^1)^\pm$ is \mathbb{G}_m -equivariant which defines, by Lemma 1.1 ii) and Example 1.2 ii), a transformation

$$p^\pm: X^\pm \longrightarrow X,$$

such that $p^+ \circ i^+ = p^- \circ i^-$ is the inclusion of the subfunctor $X^0 \subset X$.

Example 1.13. Let $X = \mathbb{P}_S^1$ as in Example 1.5. Then $i^\pm: \{0_S\} \amalg \{\infty_S\} \rightarrow X^\pm$ is the inclusion and in particular closed. The morphism $p^\pm: X^\pm \rightarrow \{0_S\} \amalg \{\infty_S\}$ is given by contracting the \mathbb{A}^1 -components of X^\pm . The morphism $q^\pm: X^\pm \rightarrow \mathbb{P}_S^1$ is the inclusion and in particular a monomorphism (but not locally closed).

Definition 1.14. Let X/S be a space with a \mathbb{G}_m -action. The commutative diagram $\text{HypLoc}(X)$ of transformations

$$\begin{array}{ccccc}
 X^0 & & & & \\
 & \searrow^{i^-} & & & \\
 & & X^+ \times_X X^- & \xrightarrow{\quad} & X^- \xrightarrow{q^-} X^0 \\
 & \searrow^j & \downarrow p^+ & \searrow p^- & \downarrow p^- \\
 & & X^+ & \xrightarrow{p^+} & X \\
 & \searrow^{i^+} & \downarrow q^+ & & \\
 & & X^0 & &
 \end{array}$$

³Informally speaking, a point $x \in X^\pm$ maps to its limit $\lim_{\lambda \rightarrow 0} \lambda \cdot x$ (resp. $\lim_{\lambda \rightarrow \infty} \lambda \cdot x$).

is called the hyperbolic localization diagram.

Remark 1.15. In view of the explicit description in Lemma 1.9, the map j is an isomorphism if X is S -affine. In general, $X^+ \times_X X^-$ is strictly bigger, e.g. for $X = \mathbb{P}_S^1$. See Proposition 1.17 iii) below for the basic property of the map.

In view of Theorem 1.8 and the definitions, we obtain the following corollary.

Corollary 1.16. *Let S be a scheme, and let X/S be a space with an étale locally linearizable \mathbb{G}_m -action. For a morphism of schemes $S' \rightarrow S$, the induced \mathbb{G}_m -action on $X' = X_{S'}$ is again étale locally linearizable. The transformation constructed in Lemma 1.1 iii) defines a \mathbb{G}_m -equivariant isomorphism of commutative diagrams of S' -spaces*

$$\mathrm{HypLoc}(X') \xrightarrow{\sim} \mathrm{HypLoc}(X) \times S'.$$

□

Let us mention some basic properties of the morphisms appearing in $\mathrm{HypLoc}(X)$.

Proposition 1.17. *Let S be a scheme, and let X/S be a space with an étale locally linearizable \mathbb{G}_m -action. The morphism of S -spaces*

- i) $i^\pm: X^0 \rightarrow X^\pm$ is a closed immersion;
- ii) $q^\pm: X^\pm \rightarrow X^0$ is affine, has geometrically connected fibers and induces a bijection on connected components;
- iii) $j = (i^+, i^-): X^0 \rightarrow X^+ \times_X X^-$ is an open and closed immersion.

Remark 1.18. Part iii) is [Dr13, Prop. 1.6.2] over fields.

Proof. Part i) follows from the fact that $p^+ \circ i^+ = p^- \circ i^-$ is the inclusion of the closed subspace $X^0 \subset X$. Part ii) is Corollary 1.12. Consider part iii). Clearly, j is a closed immersion, and we show that it is also étale. Let $\{U_i \rightarrow X\}_i$ be a S -affine \mathbb{G}_m -equivariant étale covering family. Lemma 1.10 shows that the commutative diagram of S -spaces

$$(1.10) \quad \begin{array}{ccc} \coprod_i U_i^0 & \longrightarrow & X^0 \\ \downarrow & & \downarrow \\ \coprod_{i,j} (U_i^+ \times_X U_j^-) & \longrightarrow & X^+ \times_X X^- \end{array}$$

is cartesian (apply the lemma to $U_i^\pm \rightarrow X^\pm$ and use $(X^\pm)^0 = X^0$). The bottom arrow in (1.10) is étale and surjective. By descent it is enough to show that $U_i^0 \rightarrow U_i^+ \times_X U_i^-$ is étale. Since $U_i \rightarrow X$ is étale, the diagonal $U_i \rightarrow U_i \times_X U_i$ is étale, and hence the morphism $U_i^+ \times_{U_i} U_i^- \rightarrow U_i^+ \times_X U_i^-$ obtained by base change is étale. Now the explicit description in Lemma 1.9 shows that $U_i^0 = U_i^+ \times_{U_i} U_i^-$. The proposition follows. □

Remark 1.19. i) If X/S is separated, then $p^\pm: X^\pm \rightarrow X$ is a monomorphism. In general this fails, e.g. for the affine line with double origin.

ii) It is suprising that j is an open immersion even for non-normal schemes, e.g. if X is \mathbb{P}^1 with 0 and ∞ identified. Then $X^0 = \{*\}$ but $X^\pm \neq X$ as one might guess. Indeed, consider the \mathbb{G}_m -equivariant projection $\mathbb{P}^1 \rightarrow X$. It induces an isomorphism $(\mathbb{A}^1)^\pm \subset (\mathbb{P}^1)^\pm \rightarrow X^\pm$, and one gets $X^+ \times_X X^- = \mathbb{G}_m \amalg \{*\}$.

2. HYPERBOLIC LOCALIZATION IN FAMILIES

Let S be a scheme, and let X/S be a space. Let $\Lambda = \mathbb{Z}/n$ with $n > 1$ invertible on S . We denote by $D(X, \Lambda)$ the unbounded derived category of $(X_{\text{ét}}, \Lambda)$ -modules, where $X_{\text{ét}}$ denotes

the étale topos associated with X . If $f: Y \rightarrow X$ is a morphism of S -spaces, there are the Grothendieck operations

$$\begin{aligned} f^*: D(X, \Lambda) &\rightarrow D(Y, \Lambda), & f_*: D(Y, \Lambda) &\rightarrow D(X, \Lambda), \\ - \otimes_X -: D(X, \Lambda) \times D(X, \Lambda) &\longrightarrow D(X, \Lambda), \\ \mathrm{Hom}_X: D(X, \Lambda)^{\mathrm{op}} \times D(X, \Lambda) &\longrightarrow D(X, \Lambda), \end{aligned}$$

where (f^*, f_*) and $(- \otimes_X \mathcal{A}, \mathrm{Hom}_X(\mathcal{A}, -))$ for every $\mathcal{A} \in D(X, \Lambda)$ are pairs of adjoint functors. If $f: Y \rightarrow X$ is locally of finite type, there is another pair $(f_!, f^!)$ of adjoint functors

$$f_!: D(Y, \Lambda) \rightarrow D(X, \Lambda), \quad f^!: D(X, \Lambda) \rightarrow D(Y, \Lambda).$$

These operations satisfy the usual properties: base change, projection and Künneth formula, trace map and Poincaré duality, cf. the work of Liu-Zheng [LZ12] for all statements in full generality and the references cited there.

Further, if $Z \xrightarrow{i} X \xleftarrow{j} X \setminus Z$ is a closed immersion with open complement, then for any $\mathcal{A} \in D(X, \Lambda)$ there are distinguished triangles

$$\begin{aligned} j_! j^* \mathcal{A} &\longrightarrow \mathcal{A} \longrightarrow i_* i^* \mathcal{A} \longrightarrow \\ i_* i^! \mathcal{A} &\longrightarrow \mathcal{A} \longrightarrow j_* j^* \mathcal{A} \longrightarrow, \end{aligned}$$

and we have $i^* j_! = 0 = i^! j_*$, $i^* i_* \simeq \mathrm{id} \simeq i^! i_*$ and $j^* j_! \simeq \mathrm{id} \simeq j^* j_*$, cf. [StaPro, TAG 0A4L] for the second triangle.

2.1. Construction of $L_{X/S}^- \rightarrow L_{X/S}^+$. Let X/S be a space locally of finite presentation with an étale locally linearizable \mathbb{G}_m -action. By Theorem 1.8 the hyperbolic localization diagram $\mathrm{HypLoc}(X)$ in (1.14) is a commutative diagram of S -spaces locally of finite presentation. Let us define two functors by pull-push along the maps

$$\begin{array}{ccc} & X^\pm & \\ q^\pm \swarrow & & \searrow p^\pm \\ X^0 & & X. \end{array}$$

Definition 2.1. The functors $L_{X/S}^\pm: D(X, \Lambda) \rightarrow D(X^0, \Lambda)$ are defined as

$$\begin{aligned} L_{X/S}^+ &\stackrel{\mathrm{def}}{=} (q^+) \circ (p^+)^* \\ L_{X/S}^- &\stackrel{\mathrm{def}}{=} (q^-)_* \circ (p^-)^!. \end{aligned}$$

Construction 2.2. Braden [Br03] (cf. also [DG15]) constructs a transformation of functors

$$(2.1) \quad L_{X/S}^- \longrightarrow L_{X/S}^+$$

as follows. By Proposition 1.17 iii), the morphism

$$(2.2) \quad j \stackrel{\mathrm{def}}{=} (i^+, i^-): X^0 \longrightarrow X^+ \times_X X^-$$

is an open and closed immersion, hence $(j^!, j_*)$ are adjoint. Recall the notation of the maps in the definition of $\mathrm{HypLoc}(X)$, cf. Definition 1.14. Applying the functor $(i^-)^*(p^-)^!$ to the unit of the adjunction $\mathrm{id} \rightarrow (p^+)_*(p^+)^*$, we get

$$\begin{aligned} (i^-)^*(p^-)^! &\longrightarrow (i^-)^*(p^-)^!(p^+)_*(p^+)^* \\ &\simeq (i^-)^*(p^-)_*(p^+)^!(p^+)^* && \text{(base change)} \\ (2.3) \quad &\longrightarrow (i^-)^*(p^-)_* j_* j^!(p^+)^!(p^+)^* && ((j^!, j_*)\text{-adjunction}) \\ &\simeq (i^+)^!(p^+)^*. && ((p^\pm) \circ j = i^\pm \text{ and } (i^-)^*(i^-)_* \simeq \mathrm{id}) \end{aligned}$$

Now precompose (resp. compose) (2.3) with the transformation

$$(2.4) \quad (q^-)_* \longrightarrow (i^-)^* \quad (\text{resp. } (i^+)^! \longrightarrow (q^+)_!)$$

obtained from the unit $\text{id} \rightarrow (i^-)_*(i^-)^*$ (resp. the counit $(i^+)_!(i^+)^! \rightarrow \text{id}$) by applying $(q^-)_*$ (resp. $(q^+)_!$) and using $q^\pm \circ i^\pm = \text{id}$. This constructs (2.1).

2.2. Monodromic complexes. Let $a, p: \mathbb{G}_{m,S} \times X \rightarrow X$ be the action (resp. projection) map. We call a complex $\mathcal{A} \in D(X, \Lambda)$ (*naively*) \mathbb{G}_m -equivariant if there exists an isomorphism $a^*\mathcal{A} \simeq p^*\mathcal{A}$ in $D(\mathbb{G}_{m,S} \times X, \Lambda)$.

Definition 2.3. Let $D(X, \Lambda)^{\mathbb{G}_m\text{-mon}}$ be the full subcategory of $D(X, \Lambda)$ strongly generated⁴ by \mathbb{G}_m -equivariant complexes. The objects of $D(X, \Lambda)^{\mathbb{G}_m\text{-mon}}$ are called \mathbb{G}_m -monodromic complexes.

Remark 2.4. Note that Definition 2.3 differs from Drinfeld-Gaitsgory's definition of \mathbb{G}_m -monodromic complexes. Let us recall their definition. The quotient $\mathcal{X} = [\mathbb{G}_m \backslash X]$ is an Artin stack over S , and we denote by $D(\mathcal{X}, \Lambda)$ the unbounded derived category of $(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ -modules, where $\mathcal{X}_{\text{lis-ét}}$ denotes the lisse-étale topos associated with \mathcal{X} . Let $D_{\text{cart}}(\mathcal{X}, \Lambda)$ be the full subcategory of $D(\mathcal{X}, \Lambda)$ spanned by complexes whose cohomology sheaves are cartesian. The canonical projection $\pi: X \rightarrow \mathcal{X}$ is smooth, and hence we get a morphism of topoi $X_{\text{lis-ét}} \rightarrow \mathcal{X}_{\text{lis-ét}}$. Note that $D_{\text{cart}}(X_{\text{lis-ét}}, \Lambda) = D(X_{\text{ét}}, \Lambda)$. At the level of derived categories, this gives the pullback functor

$$\pi^*: D_{\text{cart}}(\mathcal{X}, \Lambda) \rightarrow D(X, \Lambda).$$

Drinfeld-Gaitsgory define the category of \mathbb{G}_m -monodromic complexes to be the full subcategory strongly generated by the essential image of π^* . Note that each complex of the form $\pi^*\mathcal{A}$ admits a natural isomorphism

$$a^*(\pi^*\mathcal{A}) \xrightarrow{\sim} p^*(\pi^*\mathcal{A})$$

in $D(\mathbb{G}_{m,S} \times X, \Lambda)$. Hence, each \mathbb{G}_m -monodromic complex in the sense of Drinfeld-Gaitsgory is \mathbb{G}_m -monodromic in the sense of Definition 2.3.

Lemma 2.5. *Let $f: Y \rightarrow X$ be a \mathbb{G}_m -equivariant morphism of S -spaces.*

- i) The pair (f^*, f_*) restricts to a pair of adjoint functors on \mathbb{G}_m -monodromic complexes.*
- ii) If f is locally of finite type, the pair $(f_!, f^!)$ restricts to a pair of adjoint functors on \mathbb{G}_m -monodromic complexes.*

Proof. By \mathbb{G}_m -equivariance we get a cartesian diagram of S -spaces

$$\begin{array}{ccc} \mathbb{G}_{m,S} \times Y & \xrightarrow{f} & \mathbb{G}_{m,S} \times X \\ a \downarrow & & \downarrow a \\ Y & \xrightarrow{f} & X \end{array}$$

where a denotes the action (and the same cartesian diagram for the projection p). Since a and p are smooth, the lemma follows from smooth base change for equivariant complexes. The general case follows by induction. \square

2.3. Statement of Braden's theorem. Let X/S be space, and let $D(X, \Lambda)$ as above the unbounded derived category of $(X_{\text{ét}}, \Lambda)$ -modules with $\Lambda = \mathbb{Z}/n$ for some $n > 1$ invertible on S . Let us denote by $D^+(X, \Lambda)$ the full subcategory of $D(X, \Lambda)$ of bounded below complexes.

Theorem 2.6. *Let S be a scheme. Let X/S be a space locally of finite presentation with an étale locally linearizable \mathbb{G}_m -action. Then, for $\mathcal{A} \in D^+(X, \Lambda)^{\mathbb{G}_m\text{-mon}}$, the arrow of $D(X^0, \Lambda)$ defined in (2.1)*

$$L_{X/S}^-\mathcal{A} \longrightarrow L_{X/S}^+\mathcal{A}$$

is an isomorphism.

⁴Strongly generated means generated by a finite iteration of taking the cone of a morphism.

Theorem 2.6 implies Theorem B i) from the introduction. The proof follows Braden's original method: we prove the theorem for affine spaces with a linear \mathbb{G}_m -action, cf. §2.4, §2.5, §2.6 below, and reduce to the latter case using an equivariant atlas, cf. §2.7, §2.8 below. Let us warm up with the case of \mathbb{A}^1 .

2.4. Monodromic complexes on \mathbb{A}^1 . Let $q: \mathbb{A}_S^1 \rightarrow S$ denote the structure morphism. Let $i: S \rightarrow \mathbb{A}_S^1$ be any morphism of S -schemes such that $q \circ i = \text{id}$. As in (2.4) above, there are natural transformations

$$(2.5) \quad q_* \longrightarrow i^*, \quad (\text{resp. } i^! \longrightarrow q_!)$$

from $D(\mathbb{A}_S^1, \Lambda)$ to $D(S, \Lambda)$.

Lemma 2.7. *Let \mathcal{B} be a sheaf of Λ -modules on S . The transformation (2.5) is an isomorphism for $q^*\mathcal{B}$ (resp. $q^!\mathcal{B}$).*

Proof. If S is the spectrum of a separably closed field, the map

$$(2.6) \quad q_* q^*\mathcal{B} \xrightarrow{\sim} i^* q^*\mathcal{B} = \mathcal{B}$$

is an isomorphism: \mathcal{B} is of torsion invertible on S which implies that $H^i(\mathbb{A}_S^1, q^*\mathcal{B}) = 0$ for $i > 0$. In the general case, we may assume S to be the spectrum of a strictly Henselian local ring, and (2.6) follows from local acyclicity of smooth morphisms, cf. [SGA4 $\frac{1}{2}$, Arcata, §V, Thm. 1.7]⁵. In the other case, the map

$$(2.7) \quad \mathcal{B} = q^! i^! \mathcal{B} \xrightarrow{\sim} q_! q^! \mathcal{B}$$

is an isomorphism. Indeed, by [LZ12, Thm. 0.1.4 (2)], we have $q^! = q^* \langle 1 \rangle$ with $\langle 1 \rangle = [2](1)$, and base change reduces us to the case that S is the spectrum of a separably closed field. In this case, $H_c^{2-i}(\mathbb{A}_S^1, q^*\mathcal{B}(1))$ is zero for $i \neq 0$. \square

Corollary 2.8. *Let $\mathcal{B} \in D^+(\mathbb{A}_S^1, \Lambda)$. Then*

$$q_*(j_! j^* q^*\mathcal{B}) = 0 \quad (\text{resp. } q_!(j_* j^* q^!\mathcal{B}) = 0),$$

where $j: \mathbb{A}_S^1 \setminus i(S) \rightarrow \mathbb{A}_S^1$ is the inclusion.

Proof. Using the sheafy version of the Leray spectral sequence, we may assume that \mathcal{B} is a sheaf of Λ -modules. Applying q_* (resp. $q_!$) to the distinguished triangle $j_! j^*(q^*\mathcal{B}) \rightarrow q^*\mathcal{B} \rightarrow i_* i^*(q^*\mathcal{B}) \rightarrow$ (resp. $i_! i^!(q^!\mathcal{B}) \rightarrow q^!\mathcal{B} \rightarrow j_* j^*(q^!\mathcal{B}) \rightarrow$) we see that the corollary follows from Lemma 2.7. \square

Let $i_0: S \rightarrow \mathbb{A}_S^1$ denote the zero section.

Corollary 2.9. *Let $\mathcal{A} \in D^+(\mathbb{A}_S^1, \Lambda)^{\mathbb{G}_m\text{-mon}}$ where \mathbb{A}_S^1 is equipped with the standard action. The transformation $q_* \rightarrow i_0^*$ (resp. $i_0^! \rightarrow q_!$) is an isomorphism for \mathcal{A} .*

Proof. By induction we may assume that \mathcal{A} is \mathbb{G}_m -equivariant. Let $j: \mathbb{G}_{m,S} \rightarrow \mathbb{A}_S^1$ be the inclusion. Then the complex $j^*\mathcal{A}$ is of the form $j^* q^*\mathcal{B}$ for a complex $\mathcal{B} \in D^+(S, \Lambda)$. Hence, Corollary 2.8 implies the claim. \square

Let $i_1: S \rightarrow \mathbb{A}_S^1$ be the unit section.

Corollary 2.10. *Let $\mathcal{B} \in D^+(S, \Lambda)$. Let $\varphi: q^*\mathcal{B} \rightarrow q^*\mathcal{B}$ (resp. $q^!\mathcal{B} \rightarrow q^!\mathcal{B}$) be a map such that $i_1^* \varphi$ (resp. $i_1^! \varphi$) is an isomorphism and $i_0^* \varphi$ (resp. $i_0^! \varphi$) is zero. Then $\mathcal{B} = 0$.*

⁵The noetherian assumption in [SGA4 $\frac{1}{2}$] can be removed as follows. If $S = \lim_i S_i$ with S_i noetherian and affine, then on constructible sheaves $\text{Sh}_c(S) = \text{colim}_i \text{Sh}_c(S_i)$ by [SGA 4, IX Cor. 2.7.4]. By compatibility with filtered colimits (2.6) holds for constructible sheaves with S arbitrary. As every torsion sheaf is a colimit of constructible sheaves, we obtain (2.6) in general.

Proof. Since $q^*\mathcal{B}$ is \mathbb{G}_m -monodromic, there is a commutative (up to natural isomorphism) diagram in $D^+(S, \Lambda)$

$$(2.8) \quad \begin{array}{ccccc} i_1^* q^* \mathcal{B} & \xleftarrow{\simeq} & q_* q^* \mathcal{B} & \xrightarrow{\simeq} & i_0^* q^* \mathcal{B} \\ \simeq \downarrow & & \downarrow & & \downarrow 0 \\ i_1^* q^* \mathcal{B} & \xleftarrow{\simeq} & q_* q^* \mathcal{B} & \xrightarrow{\simeq} & i_0^* q^* \mathcal{B}, \end{array}$$

where the vertical maps are induced by φ . Hence, the zero map on \mathcal{B} is an isomorphism, i.e. $\mathcal{B} = 0$. The !-case follows similarly. \square

2.5. Braden's contraction lemma. The core of Theorem 2.6 for affine spaces with a linear \mathbb{G}_m -action is an analogue of Braden's contraction lemma [Br03, Lem. 6]. We follow his method which is based on arguments of Springer [Sp84].

Let \mathcal{E} be a locally free \mathcal{O}_S -module of finite rank. Let $\mathbb{V}(\mathcal{E}) = \text{Spec}_S(\text{Sym}^\otimes(\mathcal{E}))$ be the associated vector bundle over S . A linear \mathbb{G}_m -action on $\mathbb{V}(\mathcal{E})$ is equivalent to a morphism of group schemes $\mathbb{G}_{m,S} \rightarrow \text{GL}(\mathcal{E})$. If S is connected such a morphism corresponds to a \mathbb{Z} -grading on \mathcal{E} , i.e. as \mathcal{O}_S -modules

$$(2.9) \quad \mathcal{E} = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i,$$

where the decomposition is according to the weights of the \mathbb{G}_m -action. Let us fix a non-trivial decomposition

$$(2.10) \quad \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-,$$

such that $\mathcal{E}^+ = \bigoplus_{i \geq k} \mathcal{E}_i$ and $\mathcal{E}^- = \bigoplus_{i < k} \mathcal{E}_i$ for some fixed $k \in \mathbb{Z}$. Let $\mathbb{P}(\mathcal{E}) = \text{Proj}_S(\text{Sym}^\otimes(\mathcal{E}))$ be the corresponding projective bundle over S . Then $Z = \mathbb{P}(\mathcal{E}^-)$, and $Y = \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{E}^+)$ are S -schemes equipped with \mathbb{G}_m -actions. The decomposition (2.10) gives \mathbb{G}_m -equivariant maps

$$(2.11) \quad Z \xrightarrow{\iota} Y \xrightarrow{\pi} Z$$

with $\pi \circ \iota = \text{id}$ and $Z^0 = Y^0$. Note that the last equality already follows from all weights of \mathcal{E}^+ being different from all weights of \mathcal{E}^- .

Remark 2.11. Recall that $\mathbb{P}(\mathcal{E})$ represents the functor that to a S -scheme T associates the set of locally direct summand quasi-coherent \mathcal{O}_T -submodules $\mathcal{F} \subset \mathcal{E}_T$ such that $\mathcal{E}_T/\mathcal{F}$ is locally free of rank 1. In this description the map ι is given by $\mathcal{F} \mapsto \mathcal{F} \oplus \mathcal{E}_T^+$ and π is given by $\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{E}_T^-$. To check that π is well-defined note that for any locally direct summand $\mathcal{F} \subset \mathcal{E}_T$ as above not containing \mathcal{E}_T^- the natural map $\mathcal{E}_T^- \rightarrow \mathcal{E}_T/\mathcal{F}$ is surjective (use Nakayama) and gives an isomorphism $\mathcal{E}_T^-/(\mathcal{E}_T^- \cap \mathcal{F}) \simeq \mathcal{E}_T/\mathcal{F}$.

Let us denote by $\tau: Z \rightarrow S$ the structure morphism. Applying τ_* (resp. $\tau_!$) to the natural transformation $\pi_* \rightarrow \iota^*$ (resp. $\iota^! \rightarrow \pi_!$) gives

$$(2.12) \quad \tau_* \pi_* \longrightarrow \tau_* \iota^* \quad (\text{resp. } \tau_! \iota^! \rightarrow \tau_! \pi_!)$$

as natural transformations from $D(Y, \Lambda)$ to $D(S, \Lambda)$. Note that τ is proper and thus $\tau_* = \tau_!$.

Proposition 2.12 (Braden's contraction lemma). *The transformations (2.12) restricted to the category $D^+(Y, \Lambda)^{\mathbb{G}_m\text{-mon}}$ are isomorphisms.*

Remark 2.13. Let us point out that the proposition also holds true with the roles of \mathcal{E}^+ and \mathcal{E}^- interchanged. Indeed, any complex is \mathbb{G}_m -monodromic if and only if it is \mathbb{G}_m -monodromic for the inverse \mathbb{G}_m -action.

We first prove the proposition for $\tau_* \pi_* \longrightarrow \tau_* \iota^*$, and then explain the adjustments for the !-case in Remark 2.17 below. Of course, in the presence of duality both transformations are dual to each other. Let us start.

Let $\mathbb{G}_{m,S} \times Y \rightarrow \mathbb{G}_{m,S} \times Y \times Y$, $(\lambda, y) \mapsto (\lambda, y, \lambda \cdot y)$ be the graph of the action map (closed because Y/S is separated), and denote by Γ its scheme theoretic closure in $\mathbb{A}_S^1 \times Y \times Y$. Let $p_1: \Gamma \rightarrow \mathbb{A}_S^1 \times Y$, $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ and let $p_2: \Gamma \rightarrow \mathbb{A}_S^1 \times Y$, $(y_1, y_2, y_3) \mapsto (y_1, y_3)$.

Lemma 2.14. *i) The morphisms $p_1, p_2: \Gamma \rightarrow \mathbb{A}_S^1 \times Y$ are isomorphisms over $\mathbb{G}_{m,S} \times Y$.
ii) The morphism $p_1: \Gamma \rightarrow \mathbb{A}_S^1 \times Y$ is proper.*

Remark 2.15. For part ii), the hypothesis on the weights in (2.10) is crucial.

Proof. Part i) follows from the following elementary fact. If $I \subset R[t, t^{-1}]$ is an ideal for some ring R , then $I = (I \cap R[t]) \cdot R[t, t^{-1}]$ by flatness of localizations. We claim that Γ is a closed subscheme of $\mathbb{A}_S^1 \times Y \times \mathbb{P}(\mathcal{E})$. Consider the cartesian diagram of S -schemes

$$\begin{array}{ccc} \mathbb{G}_{m,S} \times Y \times Y & \longrightarrow & \mathbb{G}_{m,S} \times Y \times \mathbb{P}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathbb{A}_S^1 \times Y \times Y & \longrightarrow & \mathbb{A}_S^1 \times Y \times \mathbb{P}(\mathcal{E}), \end{array}$$

where all maps are open immersions. Let Γ_a denote the graph of the action map $a: \mathbb{G}_{m,S} \times Y \rightarrow Y$. Then Γ_a is a closed subscheme of both $\mathbb{G}_{m,S} \times Y \times Y$ and $\mathbb{G}_{m,S} \times Y \times \mathbb{P}(\mathcal{E})$ (follows from the surjectivity of the multiplication map $A \otimes_R A[f^{-1}] \rightarrow A[f^{-1}]$ for a ring A and any $f \in A$). Let $\tilde{\Gamma}$ be the closure of Γ_a in $\mathbb{A}_S^1 \times Y \times \mathbb{P}(\mathcal{E})$. Since $\mathbb{G}_{m,S} \subset \mathbb{A}_S^1$ is quasi-compact, we have by [StaPro, Tag 01R5, Lem. 28.6.3] that $\Gamma = \tilde{\Gamma} \cap (\mathbb{A}_S^1 \times Y \times Y)$. Hence, $\Gamma \rightarrow \tilde{\Gamma}$ is an open immersion, and it is enough to show that $|\Gamma| \rightarrow |\tilde{\Gamma}|$ is surjective on topological spaces. For this we may assume S to be the spectrum of an algebraically closed field. It is enough to show that $|\Gamma| \rightarrow |\tilde{\Gamma}|$ is surjective over points lying in the zero section of \mathbb{A}_S^1 . An argument on coordinates implies that whenever $[y^+ : y^-] \in Y(S)$ (i.e. $y^- \neq 0$) is a point, then its limit $\lim_{\lambda \rightarrow 0} \lambda \cdot [y^+ : y^-]$ in $\mathbb{P}(\mathcal{E})(S)$ is of the form $[0 : 'y^-] \in Y(S)$ (because of the weight hypothesis in (2.10)). This shows surjectivity of $|\Gamma| \rightarrow |\tilde{\Gamma}|$ and proves the lemma. \square

Consider the following diagram of S -schemes with the square being cartesian

$$(2.13) \quad \begin{array}{ccc} \Gamma & \xrightarrow[p_2]{p_1} & \mathbb{A}_S^1 \times Y \\ & \swarrow q_1 & \searrow q_2 \\ \mathbb{A}_S^1 & & Y \\ & \searrow q & \swarrow \tau \circ \pi \\ & S & \end{array}$$

where q_1, q_2 denote the projections. Let $Z \xrightarrow{\iota} Y \xleftarrow{\sigma} U$ where $U = Y/Z$ is the open complement.

Lemma 2.16. *Let $\mathcal{A} \in D^+(Y, \Lambda)$ be \mathbb{G}_m -equivariant. Then there is an isomorphism in $D^+(\mathbb{A}_S^1, \Lambda)$*

$$q_{1,*} p_{2,*} p_2^* q_2^* (\sigma_! \sigma^* \mathcal{A}) \simeq j_! j^* q^* \tau_* \pi_* (\sigma_! \sigma^* \mathcal{A})$$

where $j: \mathbb{G}_{m,S} \rightarrow \mathbb{A}_S^1$ denotes the inclusion.

Proof. Let $\tilde{\mathcal{A}} = q_{1,*} p_{2,*} p_2^* q_2^* (\sigma_! \sigma^* \mathcal{A})$. By smooth base change $j^* \tilde{\mathcal{A}} \simeq j^* q^* \tau_* \pi_* (\sigma_! \sigma^* \mathcal{A})$ because p_2 is an isomorphism over $\mathbb{G}_{m,S} \times Y$ by Lemma 2.14 i). Let $i: S \rightarrow \mathbb{A}_S^1$ be the zero section. By considering the distinguished triangle $j_! j^* \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}} \rightarrow i_* i^* \tilde{\mathcal{A}} \rightarrow$, we are reduced to show $i^* \tilde{\mathcal{A}} = 0$. Since $q_1 \circ p_1 = q_1 \circ p_2$, we get

$$(2.14) \quad \tilde{\mathcal{A}} \simeq q_{1,*} p_{1,!} p_2^* q_2^* (\sigma_! \sigma^* \mathcal{A})$$

because p_1 is proper, cf. Lemma 2.14 ii). There is a cartesian diagram of S -schemes

$$(2.15) \quad \begin{array}{ccc} \mathbb{G}_{m,S} \times U & \xrightarrow{\sigma'} & \Gamma \\ a \downarrow & & \downarrow q_2 \circ p_2 \\ U & \xrightarrow{\sigma} & Y, \end{array}$$

where $\sigma': \mathbb{G}_{m,S} \times U \rightarrow \Gamma$, $(\lambda, u) \mapsto (\lambda, u, \lambda \cdot u)$ and $a: \mathbb{G}_{m,S} \times U \rightarrow U$, $(\lambda, u) \mapsto \lambda \cdot u$ is the action. By (2.14), the complex $\tilde{\mathcal{A}}$ becomes

$$(2.16) \quad q_{1,*}p_{1,!}(\sigma'_! q_2^* \sigma^* \mathcal{A}) \simeq q_{1,*}p_{1,!}(\sigma'_! a^*) \sigma^* \mathcal{A} \simeq q_{1,*}p_{1,!} \sigma'_! (a^* \sigma^* \mathcal{A}).$$

Since \mathcal{A} and hence $\sigma^* \mathcal{A}$ are \mathbb{G}_m -equivariant, we have $a^*(\sigma^* \mathcal{A}) \simeq q_2^*(\sigma^* \mathcal{A})$. Moreover, $p_1 \circ \sigma' = j \times \sigma$, and (2.16) becomes

$$(2.17) \quad q_{1,*}p_{1,!} \sigma'_! (q_2^* \sigma^* \mathcal{A}) \simeq q_{1,*}(j \times \sigma)_! q_2^* \sigma^* \mathcal{A} \simeq q_{1,*}j_! j^* (q_2^* \sigma^* \mathcal{A})$$

where $j: \mathbb{G}_{m,S} \times Y \rightarrow \mathbb{A}_S^1 \times Y$ denotes the inclusion (by abuse of notation). By Corollary 2.9⁶ and (2.17) we get

$$(2.18) \quad i^* \tilde{\mathcal{A}} \simeq i^* q_{1,*}j_! j^* (q_2^* \sigma^* \mathcal{A}) \simeq q_* q_{1,*}j_! j^* (q_2^* \sigma^* \mathcal{A})$$

because $q_{1,*}j_! j^* (q_2^* \sigma^* \mathcal{A})$ is \mathbb{G}_m -monodromic for the standard action on \mathbb{A}_S^1 . Since $q \circ q_1 = (\tau \circ \pi) \circ q_2$, we get for (2.18) that

$$i^* \tilde{\mathcal{A}} \simeq (\tau \circ \pi)_* q_{2,*}j_! j^* (q_2^* \sigma^* \mathcal{A}) = 0.$$

because $q_{2,*}j_! j^* (q_2^* \sigma^* \mathcal{A}) = 0$ by Corollary 2.8. □

Proof of Braden's contraction lemma. Let $\mathcal{A} \in D^+(Y, \Lambda)^{\mathbb{G}_m\text{-mon}}$. By induction we may assume that \mathcal{A} is \mathbb{G}_m -equivariant. Let $Z \xrightarrow{\iota} Y \xleftarrow{\tau} U$ as above, and apply $\tau_* \pi_*$ to the associated distinguished triangle

$$\sigma_! \sigma^* \mathcal{A} \rightarrow \mathcal{A} \rightarrow \iota_* \iota^* \mathcal{A} \rightarrow .$$

We have to show that $\mathcal{B} = \tau_* \pi_*(\sigma_! \sigma^* \mathcal{A})$ vanishes. Let $q: \mathbb{A}_S^1 \rightarrow S$ be the structure morphism. Our aim is to construct a morphism

$$\varphi: q^* \mathcal{B} \rightarrow q^* \mathcal{B}$$

in $D^+(\mathbb{A}_S^1, \Lambda)$ which is an isomorphism when restricted to the unit section and zero when restricted to the zero section. Then Corollary 2.10 implies that $\mathcal{B} = 0$. Let us construct φ . Note that $q^* \mathcal{B} \simeq q_{1,*}q_2^*(\sigma_! \sigma^* \mathcal{A})$ by smooth base change applied to (2.13). The unit $\text{id} \rightarrow p_{2,*}p_2^*$ gives a transformation

$$(2.19) \quad q_{1,*}q_2^*(\sigma_! \sigma^* \mathcal{A}) \rightarrow q_{1,*}p_{2,*}p_2^*q_2^*(\sigma_! \sigma^* \mathcal{A}).$$

Since p_2 is an isomorphism over $\mathbb{G}_{m,S} \times Y$ (cf. Lemma 2.14), it follows that (2.19) is an isomorphism restricted to $\mathbb{G}_{m,S}$. By Lemma 2.16, we have

$$q_{1,*}p_{2,*}p_2^*q_2^*(\sigma_! \sigma^* \mathcal{A}) \simeq j_! j^* q^* \tau_* \pi_*(\sigma_! \sigma^* \mathcal{A}) = j_! j^* q^* \mathcal{B},$$

where $j: \mathbb{G}_{m,S} \rightarrow \mathbb{A}_S^1$ is the inclusion. Composing (2.19) with the adjunction $j_! j^* q^* \mathcal{B} \rightarrow q^* \mathcal{B}$ constructs the desired morphism φ . This proves the proposition. □

Remark 2.17. The adjustments for the $!$ -case are as follows. By considering the triangle $\iota_! \iota^* \mathcal{A} \rightarrow \mathcal{A} \rightarrow \sigma_* \sigma^* \mathcal{A} \rightarrow$ it is enough to show that $\mathcal{B} = \tau_! \pi_!(\sigma_* \sigma^* \mathcal{A})$ vanishes. The aim is to construct a map $\varphi: q^! \mathcal{B} \rightarrow q^! \mathcal{B}$ such that $j^* \varphi$ is an isomorphism and φ vanishes when $!$ -restricted to the zero section. Then Corollary 2.10 implies that $\mathcal{B} = 0$. As above $q^! \mathcal{B} \simeq q_{1,!}q_2^!(\sigma_* \sigma^* \mathcal{A})$ by smooth base change. Now the counit $p_{2,!}p_2^! \rightarrow \text{id}$ gives a map

$$(2.20) \quad q_{1,!}p_{2,!}p_2^!q_2^!(\sigma_* \sigma^* \mathcal{A}) \rightarrow q_{1,!}q_2^!(\sigma_* \sigma^* \mathcal{A}),$$

⁶Apply the corollary to the \mathbb{G}_m -structure with respect to \mathbb{A}_S^1 ignoring the action on Y .

and as in Lemma 2.16, one shows that $q_1!p_2!p_2^!q_2^!(\sigma_*\sigma^*\mathcal{A}) \simeq j_*j^*q^!\mathcal{B}$. Precomposing (2.20) with $q^!\mathcal{B} \rightarrow j_*j^*q^!\mathcal{B}$ constructs the desired map φ .

2.6. Linear actions. Our argument follows Braden's argument in [Br03]. Let us explain how the contraction lemma (Proposition 2.12 above) implies Theorem 2.6 for affine spaces.

Let S be connected, and let \mathcal{E} be a locally free \mathcal{O}_S -module of finite rank with \mathbb{G}_m -action. Consider the weight decomposition

$$(2.21) \quad \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \oplus \mathcal{E}^0,$$

where $\mathcal{E}^0 = \mathcal{E}_0$ is the zero component in the weight decomposition (2.9), and $\mathcal{E}^+ = \oplus_{i>0} \mathcal{E}_i$ and $\mathcal{E}^- = \oplus_{i<0} \mathcal{E}_i$. By the explicit description in §1.3, the hyperbolic localization diagram $\text{HypLoc}(\mathbb{V}(\mathcal{E}))$ becomes⁷

$$(2.22) \quad \begin{array}{ccc} \mathbb{V}(\mathcal{E}^0) & \xrightarrow{i^-} & \mathbb{V}(\mathcal{E}^- \oplus \mathcal{E}^0) \xrightarrow{q^-} \mathbb{V}(\mathcal{E}^0) \\ i^+ \downarrow & & \downarrow p^- \\ \mathbb{V}(\mathcal{E}^+ \oplus \mathcal{E}^0) & \xrightarrow{p^+} & \mathbb{V}(\mathcal{E}) \\ q^+ \downarrow & & \\ & & \mathbb{V}(\mathcal{E}^0), \end{array}$$

where all maps are induced by the decomposition (2.21).

Proposition 2.18. *Let S be a connected scheme. Then Theorem 2.6 holds for $X = \mathbb{V}(\mathcal{E})$ with a linear \mathbb{G}_m -action.*

As a benefit of working over a general base S , we may and do assume that $\mathbb{V}(\mathcal{E}^0) = S$, i.e. $\mathcal{E}^0 = 0$.

Lemma 2.19. *Let \mathcal{A} be a \mathbb{G}_m -monodromic bounded below complex of Λ -modules on $\mathbb{V}(\mathcal{E}^-)$ (resp. $\mathbb{V}(\mathcal{E}^+)$). Then the transformation*

$$(q^-)_*\mathcal{A} \xrightarrow{\simeq} (i^-)^*\mathcal{A} \quad (\text{resp. } (i^+)^!\mathcal{A} \xrightarrow{\simeq} (q^+)_!\mathcal{A})$$

is an isomorphism.

Proof. Apply Proposition 2.12 to $\mathcal{O}_S \oplus \mathcal{E}^-$ (resp. $\mathcal{E}^+ \oplus \mathcal{O}_S$) where \mathcal{O}_S is of weight 0. Then $Z = S$ and $Y = \mathbb{V}(\mathcal{E}^-)$ using the inverse \mathbb{G}_m -action (resp. $Y = \mathbb{V}(\mathcal{E}^+)$). \square

Remark 2.20. i) Similar lemmas are well known in different contexts, cf. [DG11, Prop. 5.3.2] and the references cited there.

ii) Let us sketch a direct proof of Lemma 2.19 which is independent of Proposition 2.12. Blowing up the zero section in $\mathbb{V}(\mathcal{E}^\pm)$ one reduces to the case of ${}^x\mathbb{A}_S^1$ where \mathbb{G}_m -acts through a character $\chi: \mathbb{G}_m \rightarrow \mathbb{G}_m$, $\lambda \mapsto \lambda^a$ with $a \neq 0$. Inverting the \mathbb{G}_m -action if necessary we may assume that $a > 0$. Consider the \mathbb{G}_m -equivariant finite flat map

$$\pi: \mathbb{A}_S^1 \longrightarrow {}^x\mathbb{A}_S^1, \quad x \longmapsto x^a,$$

where \mathbb{A}_S^1 is equipped with the standard action. If a is invertible on S , then $\pi|_{\mathbb{G}_m, S}$ is étale. If S is an \mathbb{F}_p -scheme and $a = p$, then π is the relative Frobenius. A case analysis shows that $q_*\mathcal{A} \simeq q_*\pi_*\pi^*\mathcal{A}$ and $q_!\mathcal{A} \simeq q_!\pi_!\pi^!\mathcal{A}$ where q denotes the structure morphism. Hence, we may reduce to \mathbb{A}_S^1 with the standard action. Then Corollary 2.9 implies the lemma.

Proof of Proposition 2.18. Let $\mathcal{A} \in D^+(\mathbb{V}(\mathcal{E}), \Lambda)^{\mathbb{G}_m\text{-mon}}$. Let $\mathbb{V}(\mathcal{E}^+) \xrightarrow{p^+} \mathbb{V}(\mathcal{E}) \xleftarrow{j} \mathbb{V}(\mathcal{E}) \setminus \mathbb{V}(\mathcal{E}^+)$, and consider the associated distinguished triangle

$$(2.23) \quad j_!j^*\mathcal{A} \longrightarrow \mathcal{A} \longrightarrow (p^+)_*(p^+)^*\mathcal{A} \longrightarrow .$$

⁷Note that $j: \mathbb{V}(\mathcal{E}^0) \rightarrow \mathbb{V}(\mathcal{E}^+ \oplus \mathcal{E}^0) \times_{\mathbb{V}(\mathcal{E})} \mathbb{V}(\mathcal{E}^- \oplus \mathcal{E}^0)$ is an isomorphism in this case.

Applying $(i^-)^*(p^-)^!$ to (2.23) the right arrow becomes

$$(2.24) \quad (i^-)^*(p^-)^!\mathcal{A} \longrightarrow (i^-)^*(p^-)^!(p^+)_*(p^+)^*\mathcal{A} \simeq (i^+)^!(p^+)^*\mathcal{A},$$

because the square in (2.22) is cartesian. By Lemma 2.19 it is enough to show that

$$(i^-)^*(p^-)^!(j_!j^*\mathcal{A}) = 0.$$

Consider the direct sum $(\mathcal{E}^+ \oplus \mathcal{O}_S) \oplus \mathcal{E}^-$ where \mathcal{O}_S is of weight 0. Let $Z = \mathbb{P}(\mathcal{E}^+ \oplus \mathcal{O}_S)$, and denote $Y = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_S) \setminus \mathbb{P}(\mathcal{E}^-)$. There is a \mathbb{G}_m -equivariant diagram of S -schemes

$$(2.25) \quad \begin{array}{ccccc} \mathbb{V}(\mathcal{E}^+) & \xrightarrow{p^+} & \mathbb{V}(\mathcal{E}) & \xleftarrow{p^-} & \mathbb{V}(\mathcal{E}^-) \\ \downarrow & & \downarrow \rho & & \searrow q^- \\ Z & \xrightarrow{\iota} & Y & \xrightarrow{\pi} & Z \xrightarrow{\tau} S, \end{array}$$

where ρ is an open immersion, and $i' = \rho \circ p^-$ is a closed immersion. Let $j': Y \setminus \mathbb{V}(\mathcal{E}^-) \rightarrow Y$ be the open complement. Let $\mathcal{B} = \rho_!(j_!j^*\mathcal{A})$, and consider the distinguished triangle

$$(2.26) \quad i'_*(i')^!\mathcal{B} \longrightarrow \mathcal{B} \longrightarrow j'_*(j')^*\mathcal{B} \longrightarrow .$$

Applying $(\tau \circ \pi)_*$ the first term in (2.26) becomes

$$(\tau \circ \pi)_* i'_*(i')^!\mathcal{B} \simeq \underbrace{(\tau \circ \pi \circ i')^!}_{=q^-} (p^-)^! \underbrace{\rho^! \rho_!}_{\simeq \text{id}} (j_!j^*\mathcal{A}) \simeq (q^-)_*(p^-)^!(j_!j^*\mathcal{A})$$

which is $(i^-)^*(p^-)^!(j_!j^*\mathcal{A})$ by Lemma 2.19. Let us show that $(\tau \circ \pi)_*$ of the second and third term in (2.26) vanishes.

(1) $(\tau \circ \pi)_*\mathcal{B} = 0$. There is a \mathbb{G}_m -equivariant commutative diagram of S -schemes

$$\begin{array}{ccccc} \mathbb{V}(\mathcal{E}^+) & \xrightarrow{p^+} & \mathbb{V}(\mathcal{E}) & \xleftarrow{j} & \mathbb{V}(\mathcal{E}) \setminus \mathbb{V}(\mathcal{E}^+) \\ \downarrow & & \downarrow \rho & & \downarrow \rho \\ Z & \xrightarrow{\iota} & Y & \xleftarrow{\sigma} & Y \setminus Z. \end{array}$$

Hence, $\mathcal{B} = \rho_!j_!j^*\mathcal{A} \simeq \sigma_!(\rho_!j^*\mathcal{A})$. This gives

$$\tau_*\pi_*\mathcal{B} \simeq \tau_*\pi_*\sigma_!(\rho_!j^*\mathcal{A}) \simeq \tau_*\iota^*\sigma_!(\rho_!j^*\mathcal{A}) = 0,$$

where we apply Proposition 2.16 to the \mathbb{G}_m -monodromic complex $\sigma_!(\rho_!j^*\mathcal{A})$ with respect to the inverse \mathbb{G}_m -action, cf. Remark 2.13. This shows (1).

(2) $(\tau \circ \pi)_*(j'_*(j')^*\mathcal{B}) = 0$. There is a \mathbb{G}_m -equivariant commutative diagram of S -schemes

$$\begin{array}{ccccc} \emptyset & \longrightarrow & \mathbb{V}(\mathcal{E}) \setminus \mathbb{V}(\mathcal{E}^+) & \xleftarrow{j'} & \mathbb{V}(\mathcal{E}) \setminus (\mathbb{V}(\mathcal{E}^+) \cup \mathbb{V}(\mathcal{E}^-)) \\ \downarrow & & \downarrow \rho \circ j & & \downarrow \rho \circ j \\ Z & \xrightarrow{\iota} & Y & \xleftarrow{j'} & Y \setminus \mathbb{V}(\mathcal{E}^-). \end{array}$$

where the squares are cartesian. Hence, $(j')^*\mathcal{B} = (j')^*(\rho \circ j)_!j^*\mathcal{A} \simeq (\rho \circ j)_!((j')^*j^*\mathcal{A})$ by base change. Now $Y \setminus \mathbb{V}(\mathcal{E}^-) = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_S) \setminus \mathbb{P}(\mathcal{E}^- \oplus \mathcal{O}_S)$, and we may consider

$$\mathbb{P}(\mathcal{E}^+) \xrightarrow{\iota'} Y \setminus \mathbb{V}(\mathcal{E}^-) \xrightarrow{\pi'} \mathbb{P}(\mathcal{E}^+) \xrightarrow{\tau'} S.$$

Now $\tau \circ \pi \circ j' = \tau' \circ \pi'$ which gives

$$\tau_*\pi_*j'_*(j')^*\mathcal{B} \simeq \tau'_*\pi'_*(\rho \circ j)_!((j')^*j^*\mathcal{A}) \simeq \tau'_*(\iota')^*(\rho \circ j)_!((j')^*j^*\mathcal{A}) = 0,$$

where we apply Proposition 2.16 to the decomposition $\mathcal{E}^+ \oplus (\mathcal{O}_S \oplus \mathcal{E}^-)$ and the \mathbb{G}_m -monodromic complex $(\rho \circ j)_!((j')^*j^*\mathcal{A})$ (again by considering the inverse of the \mathbb{G}_m -action). This shows (2) and proves the proposition. \square

2.7. The affine case. Let us explain how Proposition 2.18 implies Theorem 2.16 (Zariski locally on S) for S -affine schemes X of finite presentation with \mathbb{G}_m -action.

Lemma 2.21. *Let S be a scheme, and let X/S be a S -affine scheme of finite presentation with \mathbb{G}_m -action. Then, for some $n \geq 0$, there exists Zariski locally on S a \mathbb{G}_m -equivariant closed immersion $X \rightarrow \mathbb{A}_S^n$, where \mathbb{G}_m -acts linearly on \mathbb{A}_S^n .*

Proof. Let $S = \operatorname{Spec}(R)$ and $X = \operatorname{Spec}(B)$ be affine. If S is connected, then the assertion of the lemma is equivalent to the existence of a \mathbb{Z} -graded free R -module E of some finite rank n together with a morphism of \mathbb{Z} -graded R -algebras

$$\operatorname{Sym}^\otimes(E) \longrightarrow B,$$

which is surjective. Let $\{b_i\}_{i \in I}$ be a family of homogenous generators of the R -algebra B . Let $E = \bigoplus_{i \in I} R$, where the i -th component is given the degree $\deg(b_i)$. Then the morphism $\operatorname{Sym}^\otimes(E) \rightarrow B$ given by $(r_i)_{i \in I} \mapsto \sum_{i \in I} r_i \cdot b_i$ is surjective and \mathbb{Z} -graded. Since B is of finite type, the set of homogenous generators can be chosen to be finite. This proves the lemma for S locally connected. In general, write $S = \lim_i S_i$, where S_i is the spectrum of a finitely generated \mathbb{Z} -algebra (in particular locally connected). Since X/S is of finite presentation, it is defined over some S_i , and the lemma follows in general. \square

Since push forward under closed immersions is conservative, we are reduced to:

Lemma 2.22. *Let $f: X \rightarrow Z$ be a \mathbb{G}_m -equivariant closed immersion of S -affine schemes of finite presentation. Then there is a commutative (up to natural isomorphism) diagram of transformation of functors from $D(X, \Lambda)$ to $D(Z^0, \Lambda)$ as follows*

$$\begin{array}{ccc} (f^0)_* \circ L_{X/S}^- & \longrightarrow & (f^0)_* \circ L_{X/S}^+ \\ \simeq \downarrow & & \downarrow \simeq \\ L_{Z/S}^- \circ f_* & \longrightarrow & L_{Z/S}^+ \circ f_*, \end{array}$$

where the horizontal maps are constructed in (2.1).

The maps $X^0 \xrightarrow{i^\pm} X^\pm \xrightarrow{q^\pm} X^0$ induce by (2.4) natural transformations of functors from $D(X^\pm, \Lambda)$ to $D(S, \Lambda)$ as follows

$$(2.27) \quad (q^-)_* \longrightarrow (i^-)^* \quad \text{and} \quad (i^+)^! \longrightarrow (q^+)_!.$$

Lemma 2.23. *Let X and Z be S -affine schemes of finite presentation, and let $f: X \rightarrow Z$ be a \mathbb{G}_m -equivariant closed immersion. There are commutative (up to natural isomorphism) diagrams of transformations of functors from $D(X^\pm, \Lambda)$ to $D(Z^0, \Lambda)$*

$$\begin{array}{ccc} (q^-)_*(f^-)_* & \longrightarrow & (i^-)^*(f^-)_* \\ \simeq \downarrow & & \downarrow \simeq \\ (f^0)_*(q^-)_* & \longrightarrow & (f^0)_*(i^-)^* \end{array} \quad \begin{array}{ccc} (i^+)^!(f^+)_* & \longrightarrow & (q^+)_!(f^+)_* \\ \simeq \downarrow & & \downarrow \simeq \\ (f^0)_*(i^+)^! & \longrightarrow & (f^0)_*(q^+)_!, \end{array}$$

where the horizontal arrows are constructed from (2.27).

Proof. By the explicit description in Lemma 1.9 the maps f^0 and f^\pm are closed immersions, and we have $Z^0 = Z^\pm \times_{X^\pm} X^0$. The vertical maps are constructed from proper base change using that $f_*^\pm = f_!^\pm$. The commutativity of the functor diagrams is straight forward and left to the reader. \square

Proof of Lemma 2.22. In view of Lemma 2.23, it is enough to show that f_* commutes with the map $(i^-)^*(p^-)^! \rightarrow (i^+)^!(p^+)^*$ constructed in (2.3). There is a commutative diagram of

S -schemes

$$(2.28) \quad \begin{array}{ccccc} X^0 & \longrightarrow & X^- & & \\ & \searrow & \downarrow & \searrow & \\ & & Z^0 & \longrightarrow & Z^- \\ & \downarrow & \downarrow & \downarrow & \\ X^+ & \xrightarrow{\quad \quad} & X & \xrightarrow{\quad \quad} & Z \\ & \searrow & \downarrow & \searrow & \\ & & Z^+ & \longrightarrow & Z \end{array}$$

Since both X and Z are affine, the explicit description in Lemma 1.9 shows that all arrows in (2.28) are closed immersions. In particular monomorphisms which implies that all squares in (2.28) are cartesian. Again it is straight forward that the following diagram of transformations is commutative (up to natural isomorphism)

$$\begin{array}{ccc} (f^0)_*(i^-)^*(p^-)^! & \longrightarrow & (f^0)_*(i^+)^!(p^+)^* \\ \simeq \downarrow & & \downarrow \simeq \\ (i^-)^*(p^-)^! f_* & \longrightarrow & (i^+)^!(p^+)^* f_*, \end{array}$$

where the vertical maps are constructed from proper base change, cf. the Proof of Proposition 3.1 below for more details. This proves the proposition. \square

2.8. Finish of the proof of Theorem 2.6. Let S be a scheme, and let X/S be a space locally of finite presentations with an étale locally linearizable \mathbb{G}_m -action. Let $\{U_i \rightarrow X\}_i$ be a \mathbb{G}_m -equivariant S -affine étale covering family. Then $\{U_i^0 \rightarrow X^0\}_i$ is covering by Theorem 1.8 i). By Lemma 2.24 below, we reduce to the case that X is S -affine. Covering S with affine schemes so that the assertion of Lemma 2.21 holds, and using Lemma 2.24 again (for open immesrions), Theorem 2.6 follows from the previous section. It rests to show:

Lemma 2.24. *Let S be a scheme, and let X/S be a space locally of finite presentation with an étale locally linearizable \mathbb{G}_m -action. Let $f: U \rightarrow X$ be a \mathbb{G}_m -equivariant étale morphism with U being S -affine. Then there is a commutative (up to natural isomorphism) diagram of transformation of functors from $D(X, \Lambda)$ to $D(U^0, \Lambda)$ as follows*

$$\begin{array}{ccc} (f^0)^* \circ L_{X/S}^- & \longrightarrow & (f^0)^* \circ L_{X/S}^+ \\ \simeq \downarrow & & \downarrow \simeq \\ L_{U/S}^- \circ f^* & \longrightarrow & L_{U/S}^+ \circ f^*, \end{array}$$

where the horizontal maps are constructed in (2.1).

Again, let us consider the case of the natural transformations $(q^-)_* \rightarrow (i^-)^*$ and $(i^+)^! \rightarrow (q^+)_!$ first.

Lemma 2.25. *Let $f: U \rightarrow X$ as in Lemma 2.24. There are commutative (up to natural isomorphism) diagrams of transformations of functors from $D(X^\pm, \Lambda)$ to $D(U^0, \Lambda)$*

$$\begin{array}{ccc} (f^0)^*(q^-)_* & \longrightarrow & (f^0)^*(i^-)^* \\ \simeq \downarrow & & \downarrow \simeq \\ (q^-)_*(f^-)^* & \longrightarrow & (i^-)^*(f^-)^* \end{array} \quad \begin{array}{ccc} (f^0)^*(i^+)^! & \longrightarrow & (f^0)^*(q^+)_! \\ \simeq \downarrow & & \downarrow \simeq \\ (i^+)^!(f^+)^* & \longrightarrow & (q^+)_!(f^+)^*, \end{array}$$

where the horizontal arrows are constructed from (2.4).

Proof. By Lemma 1.10 we have $U^0 = U^\pm \times_{X^\pm} X^0$ (use that $(X^\pm)^0 = X^0$), and by Lemma 1.11 we have $U^\pm = U^0 \times_{X^0} X^\pm$. The diagrams of S -spaces in question are cartesian and we can use smooth base change to construct the vertical maps. Use that f^0 and f^\pm are étale and

hence, $(f^0)^* \simeq (f^0)^!$ and $(f^\pm)^* \simeq (f^\pm)^!$. The commutativity of the functor diagrams is straight forward and left to the reader. \square

Proof of Lemma 2.24. In view of Lemma 2.25 it is enough to show that f^* commutes with the transformation $(i^-)^*(p^-)^! \rightarrow (i^+)^!(p^+)^*$ in (2.1). There is a commutative diagram of S -spaces

$$\begin{array}{ccccc}
 U^0 & \xrightarrow{\quad} & U^- & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & X^0 & \xrightarrow{\quad} & X^- & \\
 & \downarrow & \downarrow & \downarrow & \\
 U^+ & \xrightarrow{\quad} & U & \xrightarrow{\quad} & X \\
 & \searrow & \downarrow & \searrow & \\
 & X^+ & \xrightarrow{\quad} & X &
 \end{array}$$

where the U -square is cartesian, and $i^\pm: X^0 \rightarrow X^\pm$ factors through $j: X^0 \rightarrow X^+ \times_X X^-$. The maps f^0 and f^\pm are étale, cf. proof of Lemma 2.25. Let us explain how one checks commutativity of the diagram

$$\begin{array}{ccc}
 (f^0)^*(i^-)^*(p^-)^! & \longrightarrow & (f^0)^*(i^+)^!(p^+)^* \\
 \simeq \downarrow & & \downarrow \simeq \\
 (i^-)^*(p^-)^! f^* & \longrightarrow & (i^+)^!(p^+)^* f^*,
 \end{array}$$

where the vertical maps are constructed using $f^* \simeq f^!$ and the same for f^0 and f^\pm . Using the units $\text{id} \rightarrow f_* f^*$ and $\text{id} \rightarrow (p^+)_*(p^+)^*$ one constructs a commutative diagram

$$\begin{array}{ccc}
 \text{id} & \longrightarrow & (p^+)_*(p^+)^* \\
 \downarrow & & \searrow \\
 f_* f^* & \longrightarrow & f_*(p^+)_*(p^+)^* f^* \xrightarrow{\simeq} (p^+)_*(f^+)_*(f^+)^*(p^+)^*.
 \end{array}$$

Using adjunction for f_* and applying $(i^-)^+(p^-)^!$ to the resulting diagram we get

$$\begin{array}{ccc}
 \underbrace{\simeq (f^0)^*(i^-)^+(p^-)^!}_{(i^-)^+(p^-)^! f^*} & \longrightarrow & \underbrace{\simeq (f^0)^*(i^-)^+(p^-)^!}_{(i^-)^+(p^-)^! f^*} (p^+)_*(p^+)^* \xrightarrow{\psi} (f^0)^*(i^+)^!(p^+)^* \\
 \downarrow & & \downarrow \simeq \\
 (i^-)^+(p^-)^! f^* & \longrightarrow & (i^-)^+(p^-)^! (p^+)_*(p^+)^* f^* \xrightarrow{\simeq} (i^+)^!(p^+)^* f^*,
 \end{array}$$

where ψ is given by the $(j^!, j_*)$ -adjunction. The composition of the arrows at the bottom (resp. the top) gives the desired map, and one checks that the right square commutes. This proves the lemma. \square

3. FUNCTORIAL PROPERTIES

As a benefit of working over a general base scheme S , we are able to investigate the behaviour of hyperbolic localization with respect to base changes $S' \rightarrow S$. The situation is as good as one could hope. This is due to strong symmetry properties induced by the isomorphism in Braden's theorem.

3.1. Base change. Recall the following formalism. If $F, F', G, G' : \mathcal{C} \rightarrow \mathcal{D}$ are functors between categories \mathcal{C} and \mathcal{D} , and $\psi : F \rightarrow G$, $\phi : F' \rightarrow G'$ are natural transformations. Then a natural 2-morphism $\Phi : \psi \Rightarrow \phi$ is a tuple $\Phi = (\Phi_F, \Phi_G)$ of natural transformations $\Phi_F : F \rightarrow F'$ and

$\Phi_G : G \rightarrow G'$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\Phi_F} & F' \\ \psi \downarrow & & \downarrow \phi \\ G & \xrightarrow{\Phi_G} & G' \end{array}$$

is commutative up to natural isomorphism. There is the obvious notion of a natural 2-isomorphism. If X/S is a space locally of finite presentation with an étale locally linearizable \mathbb{G}_m -action, then there is a natural transformation of functors on unbounded derived categories from $D(X, \Lambda)$ to $D(X^0, \Lambda)$ as follows

$$(3.1) \quad \phi_X : L_{X/S}^- \longrightarrow L_{X/S}^+.$$

We abbreviate $D(X) = D(X, \Lambda)$ and $D(X^0) = D(X^0, \Lambda)$ in the following.

Proposition 3.1. *Let S be a scheme, and let X/S be a space locally of finite presentation with an étale locally linearizable \mathbb{G}_m -action. Let $f : X' \rightarrow X$ be a \mathbb{G}_m -equivariant morphism of S -spaces. Assume that for the hyperbolic localization diagrams, cf. Definition 1.14,*

$$\text{HypLoc}(X') \xrightarrow{\cong} \text{HypLoc}(X) \times_X X'.$$

Let $f^0 : (X')^0 \rightarrow X^0$ be the induced S -morphism on the spaces of fixed points.

i) There are natural 2-morphisms as follows.

- (a) $\phi_X \circ f_* \Rightarrow f_*^0 \circ \phi_{X'}$ as 2-morphism of functors $D(X') \rightarrow D(X^0)$;*
- (b) $(f^0)^* \circ \phi_X \Rightarrow \phi_{X'} \circ f^*$ as 2-morphism of functors $D(X) \rightarrow D((X')^0)$;*

ii) If f is locally of finite presentation, then there are natural 2-morphisms as follows.

- (a) $f_!^0 \circ \phi_{X'} \Rightarrow \phi_X \circ f_!$ as 2-morphism of functors $D(X') \rightarrow D(X^0)$;*
- (b) $\phi_{X'} \circ f^! \Rightarrow (f^0)^! \circ \phi_X$ as 2-morphism of functors $D(X) \rightarrow D((X')^0)$.*

iii) If f is proper (resp. f is smooth), then i).(a) and ii).(a) (resp. i).(b) and ii).(b)) are inverse to each other.

iv) All transformations in i) and ii) restricted to the category $D^+(X)^{\mathbb{G}_m\text{-mon}}$ are natural 2-isomorphisms.

Remark 3.2. Some sort of base change hypothesis seems to be necessary in order to construct the 2-morphisms in i) and ii), cf. also Lemmas 2.24 and 2.22 above. Note that by Corollary 1.16 the base change hypothesis on $X' \rightarrow X$ is satisfied if $X' = X \times S'$ for some morphism of schemes $S' \rightarrow S$.

Proof. Recall the morphisms i^\pm, q^\pm, p^\pm in the definition of $\text{HypLoc}(X)$, cf. Definition 1.14. The corresponding morphisms for $\text{HypLoc}(X')$ are also denoted i^\pm, q^\pm, p^\pm by abuse of notation. For i).(a), we have to construct natural transformations Φ^+ and Φ^- of functors

$$(3.2) \quad \begin{array}{ccc} (q^-)_*(p^-)! f_* & \xrightarrow{\Phi^-} & (f^0)_*(q^-)_*(p^-)! \\ \phi_X f_* \downarrow & & \downarrow (f^0)_* \phi_{X'} \\ (q^+)_!(p^+)^* f_* & \xrightarrow{\Phi^+} & (f^0)_*(q^+)_!(p^+)^* \end{array}$$

such that (3.2) commutes up to natural isomorphism. By assumption both squares in the diagram

$$(3.3) \quad \begin{array}{ccccc} (X')^0 & \xleftarrow{q^\pm} & (X')^\pm & \xrightarrow{p^\pm} & X' \\ \downarrow f^0 & & \downarrow f^\pm & & \downarrow f \\ X^0 & \xleftarrow{q^\pm} & X^\pm & \xrightarrow{p^\pm} & X \end{array}$$

are cartesian.

Construction of Φ^- : Base change applied to the right square in (3.3) gives a natural isomorphism $(p^-)^\dagger f_* \simeq f_*^-(p^-)^\dagger$. Apply $(q^-)_*$ to this isomorphism, and use that the left square in (3.3) commutes. This constructs the natural isomorphism $\Phi^-: (q^-)_*(p^-)^\dagger f_* \rightarrow f_*^0(q^-)_*(p^-)^\dagger$.

Construction of Φ^+ : Applying $(p^+)^*$ to the adjunction $f^* f_* \rightarrow \text{id}$ gives

$$(f^+)^*(p^+)^* f_* \simeq (p^+)^* f^* f_* \longrightarrow (p^+)^*.$$

By adjunction, we obtain a transformation $(p^+)^* f_* \rightarrow (f^+)_*(p^+)^*$. Applying $(q^+)!$ it remains to construct a transformation

$$(3.4) \quad (q^+)!(f^+)_* \longrightarrow (f^0)_*(q^+)!.$$

Applying $(q^+)!$ to the adjunction $(f^+)^*(f^+)_* \rightarrow \text{id}$ gives

$$(f^0)^*(q^+)!(f^+)_* \simeq (q^+)!(f^+)^*(f^+)_* \longrightarrow (q^+)!,$$

where the isomorphism follows from the base change theorem applied to the left cartesian square in (3.3). Using adjunction this concludes the construction of (3.4), and hence the construction of $\Phi^+ : (q^+)!(p^+)^* f_* \rightarrow (f^0)_*(q^+)!(p^+)^*$.

Diagram (3.2) commutes up to natural isomorphism: We claim that it is enough to check the commutativity of the following diagrams, whose construction is explained below. Each isomorphism in (C1)-(C3) below is deduced by base change using our assumption $\text{HypLoc}(X') = \text{HypLoc}(X) \times_X X'$.

Compatibility 1 (C1):

$$\begin{array}{ccc} f_* & \xrightarrow{\text{id}} & f_* \\ \downarrow & & \downarrow \\ (p^+)_*(p^+)^* f_* & \longrightarrow & f_*(p^+)_*(p^+)^* \xrightarrow{\simeq} (p^+)_*(f^+)_*(p^+)^* \end{array}$$

Compatibility 2 (C2):

$$\begin{array}{ccc} (p^-)^\dagger(p^+)_*(f^+)_* & \xrightarrow{\simeq} & (f^-)_*(p^-)^\dagger(p^+)_* \\ \downarrow & & \downarrow \\ (i^-)_*(i^+)^\dagger(f^+)_* & \xrightarrow{\simeq} & (f^-)_*(i^-)_*(i^+)^\dagger \xrightarrow{\simeq} (i^-)_*(i^+)^\dagger(f^+)_* \end{array}$$

Compatibility 3 (C3):

$$\begin{array}{ccc} (i^+)^\dagger(f^+)_* & \xrightarrow{\simeq} & (f^0)_*(i^+)^\dagger \\ \downarrow & & \downarrow \\ (q^+)!(f^+)_* & \longrightarrow & (f^0)_*(q^+)! \end{array}$$

We give the recipe how (C1)-(C3) imply the commutativity of (3.2)⁸. Apply $(p^-)^\dagger$ from the left to (C1). Using base change, the upper right of (C1) may be replaced by $(f^-)_*(p^-)^\dagger$. Next apply (C2) to the lower right, and extend the resulting diagram at the very left by the commutative diagram

$$\begin{array}{ccc} (p^-)^\dagger(p^+)_*(p^+)^* f_* & \longrightarrow & (p^-)^\dagger(p^+)_*(f^+)_*(p^+)^* \\ \downarrow & & \downarrow \\ (i^-)_*(i^+)^\dagger(p^+)^* f_* & \longrightarrow & (i^-)_*(i^+)^\dagger(f^+)_*(p^+)^*, \end{array}$$

which is derived from $(p^+)^* f_* \rightarrow (f^+)_*(p^+)^*$ (cf. the construction of Φ^+) and $(p^-)^\dagger(p^+)_* \rightarrow (i^-)_*(i^+)^\dagger$ (cf. the middle arrow in (2.3)). Now apply $(q^-)_*$ to everything, and use $(q^-)_*(i^-)_* =$

⁸The author recommends a big sheet of paper to check the commutativity.

id. By base change, the morphism at the top is Φ^- . At the lower right apply (C3), and extend the resulting diagram at the very left by the commutative diagram

$$\begin{array}{ccc} (i^+)!(p^+)^*f_* & \longrightarrow & (i^+)!(f^+)_*(p^+)^* \\ \downarrow & & \downarrow \\ (q^+)!(p^+)^*f_* & \longrightarrow & (q^+)!(f^+)_*(p^+)^*. \end{array}$$

The morphism at the bottom is Φ^+ . This implies the commutativity of (3.2). It rests to show (C1)-(C3).

Proof of (C1): There is a commutative diagram

$$\begin{array}{ccc} (p^+)^*f_* & \xrightarrow{\text{id}} & (p^+)^*f_* \\ \text{id} \downarrow & & \downarrow \\ (p^+)^*f_* & \longrightarrow & (f^+)_*(p^+)_*, \end{array}$$

where $(p^+)^*f_* \rightarrow (f^+)_*(p^+)_*$ is the morphism defined in the construction of Φ^+ above. By $((p^+)^*, (p^+)_*)$ -adjunction applied to the vertical arrows and using $(p^+)_*(f^+)_* = f_*(p^+)_*$ (at the lower right), we obtain (C1).

Proof of (C2): The vertical arrows in (C2) are constructed as in (2.3) above using the $(j^!, j_*)$ -adjunction. Then (C2) follows by base change from the fact that the following diagram

$$\begin{array}{ccccc} (X')^0 & \longrightarrow & (X')^+ \times_{X'} (X')^- & \longrightarrow & X' \\ \downarrow f^0 & & \downarrow & & \downarrow f \\ X^0 & \longrightarrow & X^+ \times_X X^- & \longrightarrow & X \end{array}$$

is cartesian.

Proof of (C3): Using the adjunctions $(i^+)!(i^+)^! \rightarrow \text{id}$ and $(f^+)^*(f^+)_* \rightarrow \text{id}$, one constructs a commutative diagram

$$\begin{array}{ccc} (f^+)^*(f^+)_*(i^+)!(i^+)^! & \longrightarrow & (i^+)!(i^+)^! \\ \downarrow & & \downarrow \\ (f^+)^*(f^+)_* & \longrightarrow & \text{id}. \end{array}$$

Now apply $(q^+)!$ to the diagram. Using base change (at the left), additionally $(i^+)! = (i^+)_*$ (at the upper left) and $(q^+)!(i^+)^! = \text{id}$ (at the top), the diagram becomes

$$\begin{array}{ccc} (f^0)^*(i^+)!(f^+)_* & \longrightarrow & (i^+)^! \\ \downarrow & & \downarrow \\ (f^0)^*(q^+)!(f^+)_* & \longrightarrow & (q^+)!, \end{array}$$

and we obtain (C3) by $((f^0)^*, (f^0)_*)$ -adjunction. The bottom is the natural transformation (3.4) and the vertical arrows are deduced from $(i^+)^! \rightarrow (q^+)!$.

This proves (C1)-(C3), and hence part i).(a). The 2-morphism in part i).(b) is constructed from part i).(a) as follows. By adjunction we get a 2-morphism $(f^0)^*\phi_X f_* \Rightarrow \phi_{X'}$. Applying f^* from the right gives $(f^0)^*\phi_X f_* f^* \Rightarrow \phi_{X'} f^*$. Now define the 2-morphism in part i).(b) as the composition

$$(f^0)^*\phi_X \Rightarrow (f^0)^*\phi_X f_* f^* \Rightarrow \phi_{X'} f^*,$$

where the first arrow is deduced from the adjunction morphism $\text{id} \rightarrow f_* f^*$. Note that directly constructing $(f^0)^*\phi_X \Rightarrow \phi_{X'} f^*$ results in the same 2-morphism. This shows part i).

Part ii).(a) follows from part i).(a) by formally interchanging all $*$ with $!$, inverting all arrows and interchanging all $+$ with $-$. Part ii).(b) follows again from part ii).(a) by adjunction. This

shows part ii).

Now if f is proper (resp. smooth), then f^0 and f^\pm are proper (resp. smooth) by base change. In this case, both Φ^+ and Φ^- are deduced from proper (resp. smooth) base change and hence are isomorphisms and the corresponding transformations are inverse to each other. This shows part iii).

Let \mathcal{A} be a \mathbb{G}_m -monodromic bounded below complex. Then f_* is \mathbb{G}_m -monodromic, and both transformations $\phi_X f_*(\mathcal{A})$ and $(f^0)_* \phi_{X'}(\mathcal{A})$ in (3.2) are isomorphisms. The transformation Φ^- in (3.2) is deduced from base change, hence an isomorphism. By three out of four it follows that Φ^+ in (3.2) needs to be an isomorphism. In all other cases for f^* and $f_!$, $f^!$ (if exist) again three out of four transformations are isomorphisms and hence the remaining transformation needs to be an isomorphism. This implies part iv) and the proposition follows. \square

3.2. Commutation with nearby cycles. Let \mathcal{O} be a henselian discrete valuation ring with field of fractions F and residue field k . Let \bar{F} be a separable closure of F , and denote by $\bar{\mathcal{O}}$ the integral closure of \mathcal{O} in \bar{F} . Let \bar{k} be the residue field of $\bar{\mathcal{O}}$ (which is a separable closure of k). Let $S = \text{Spec}(\mathcal{O})$, $s = \text{Spec}(k)$, $\eta = \text{Spec}(F)$, $\bar{S} = \text{Spec}(\bar{\mathcal{O}})$, $\bar{s} = \text{Spec}(\bar{k})$, $\bar{\eta} = \text{Spec}(\bar{F})$. This gives the 7-tuple $(S, s, \eta, \bar{S}, \bar{s}, \bar{\eta}, \Gamma)$, where $\Gamma = \text{Gal}(\bar{F}/F)$ is the Galois group. For a space X/S , there is a commutative diagram

$$\begin{array}{ccccc} X_{\bar{\eta}} & \xrightarrow{\bar{j}} & X_{\bar{S}} & \xleftarrow{\bar{i}} & X_{\bar{s}} \\ \downarrow & & \downarrow & & \downarrow \\ X_{\eta} & \xrightarrow{j} & X & \xleftarrow{i} & X_s \end{array} \quad \text{cartesian above} \quad \begin{array}{ccccc} \bar{\eta} & \longrightarrow & \bar{S} & \longleftarrow & \bar{s} \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \longrightarrow & S & \longleftarrow & s. \end{array}$$

Let $\Lambda = \mathbb{Z}/n$ with $n > 1$ invertible on S . By [SGA 7, XIII], there is the functor of nearby cycles

$$\begin{aligned} \Psi_X : D(X_{\eta}, \Lambda) &\longrightarrow D(X_s \times_S \eta, \Lambda), \\ \mathcal{A} &\longmapsto \bar{i}^* \bar{j}_* \mathcal{A}_{\bar{\eta}} \end{aligned}$$

where \bar{j}_* denotes the derived push forward, and $D(X_s \times_S \eta, \Lambda)$ is as in [SGA 7, XIII] the derived category of $((X_{\bar{s}})_{\text{ét}}, \Lambda)$ -modules with a continuous action of Γ compatible its action on $X_{\bar{s}}$. For a morphism S -morphism $f: Y \rightarrow X$ we get functors f_* , f^* and $f_!$, $f^!$ (if f is locally of finite type) on the category $D(X_s \times_S \eta, \Lambda)$ as in [SGA 7, XIII 2.1.6, 2.1.7] satisfying the usual adjointness properties and functorialities with respect to Ψ .

If X/S is equipped with a \mathbb{G}_m -action, then, for any $S' \rightarrow S$, the scheme $X_{S'}$ is equipped with the induced \mathbb{G}_m -action (by base change). If the action is étale locally linearizable, there is for any $\mathcal{A} \in D(X_{S'}, \Lambda)$ the arrow of $D(X_{S'}^0, \Lambda)$ defined in (2.1)

$$L_{S'}^- \mathcal{A} \longrightarrow L_{S'}^+ \mathcal{A},$$

where $L_{S'}^\pm = L_{X_{S'}/S'}^\pm$ denote the hyperbolic localization functors, cf. Definition 2.1.

Theorem 3.3. *Let S be the spectrum of a henselian discrete valuation ring, and let X/S be a space of finite type with an étale locally linearizable \mathbb{G}_m -action. Then, for $\mathcal{A} \in D^+(X_{\eta}, \Lambda)$, there is a commutative diagram of arrows in $D(X_s \times_S \eta, \Lambda)$*

$$\begin{array}{ccc} L_{\bar{s}}^- \circ \Psi_X(\mathcal{A}) & \longleftarrow & \Psi_{X^0} \circ L_{\bar{\eta}}^-(\mathcal{A}) \\ \downarrow & & \downarrow \\ L_{\bar{s}}^+ \circ \Psi_X(\mathcal{A}) & \longrightarrow & \Psi_{X^0} \circ L_{\bar{\eta}}^+(\mathcal{A}), \end{array}$$

and all arrows are isomorphisms if \mathcal{A} is \mathbb{G}_m -monodromic.

Remark 3.4. The transformation $L_{\bar{s}}^- \rightarrow L_{\bar{s}}^+$ is defined on the category $D(X_s \times_S \eta, \Lambda)$ since its construction in (2.1) is purely formal using adjointness properties.

Proof. By Proposition 3.1 i), there is a commutative (up to natural isomorphism) diagram of transformations

$$\begin{array}{ccc} L_{\bar{S}}^- \bar{j}_* & \longrightarrow & \bar{j}_*^0 L_{\bar{\eta}}^- \\ \downarrow & & \downarrow \\ L_{\bar{S}}^+ \bar{j}_* & \longrightarrow & \bar{j}_*^0 L_{\bar{\eta}}^+. \end{array}$$

Applying $(\bar{i}^0)^*$ and using Proposition 3.1 iii), we get a commutative (up to natural isomorphism) diagram of transformations

$$(3.5) \quad \begin{array}{ccccc} L_{\bar{S}}^- \bar{i}^* \bar{j}_* & \longleftarrow & (\bar{i}^0)^* L_{\bar{S}}^- \bar{j}_* & \xrightarrow{\sim} & (\bar{i}^0)^* \bar{j}_*^0 L_{\bar{\eta}}^- \\ \downarrow & & \downarrow & & \downarrow \\ L_{\bar{S}}^+ \bar{i}^* \bar{j}_* & \xleftarrow{\sim} & (\bar{i}^0)^* L_{\bar{S}}^+ \bar{j}_* & \longrightarrow & (\bar{i}^0)^* \bar{j}_*^0 L_{\bar{\eta}}^+. \end{array}$$

Using Proposition 3.1 iii) and a limit argument we see that $L_{\bar{\eta}}^\pm \circ (-)_{\bar{\eta}} \xrightarrow{\sim} (-)_{\bar{\eta}} \circ L_{\bar{\eta}}^\pm$. This concludes the construction of the diagram above. By construction the transformations agree with the ones coming from the functorialities of the nearby cycles.

Now if \mathcal{A} is \mathbb{G}_m -monodromic, then by Lemma 2.5, the complexes $\mathcal{A}_{\bar{\eta}}$, $\bar{j}_* \mathcal{A}_{\bar{\eta}}$ and $\Psi_X \mathcal{A}$ are \mathbb{G}_m -monodromic and hence by Theorem 2.6, all vertical arrows in (3.5) are isomorphisms. This concludes the proof of the theorem. \square

Example 3.5. Let $S = \text{Spec}(\mathbb{Z}_p)$, and let X be the flat projective \mathbb{Z}_p -scheme such that $X_\eta = \mathbb{P}_\eta^1$ and such that X_s is the intersection of two \mathbb{P}_s^1 's meeting transversally at a single s -point e_s . The scheme X is equipped with a \mathbb{G}_m -action inducing the usual action on \mathbb{P}_η^1 . The \mathbb{Q}_p -points 0_η and ∞_η extend by properness to \mathbb{Z}_p -points 0_S and ∞_S which are fixed under the \mathbb{G}_m -action. Then on reduced loci $X^0 = 0_S \amalg \infty_S \amalg e_s$ is the subscheme of fixed points. The attractor (resp. repeller) is on reduced loci

$$(3.6) \quad X^+ = (\mathbb{A}_S^1)^+ \amalg \infty_S \amalg (\mathbb{A}_S^1)^+ \quad (\text{resp. } X^- = 0_S \amalg (\mathbb{A}_S^1)^- \amalg (\mathbb{A}_S^1)^-).$$

The maps $p^\pm: X^\pm \rightarrow X$ are monomorphisms (because X is separated) such that on intersections

$$X^+ \times_X X^- = X^0 \amalg \mathbb{G}_{m,\eta} \amalg \mathbb{G}_{m,s} \amalg \mathbb{G}_{m,s}.$$

The morphisms $q^\pm: X^\pm \rightarrow X^0$ are given by contracting (3.6) to the fixed points. The complex $\mathcal{A} = \mathbb{Z}/n\langle 1 \rangle$ on X_η is \mathbb{G}_m -monodromic, and one computes for the hyperbolic localization

$$L_{X_\eta/\eta}^\pm(\mathcal{A}) = \mathbb{Z}/n\langle -1 \rangle \oplus \mathbb{Z}/n\langle 1 \rangle.$$

The nearby cycles Ψ_{X^0} are constant, i.e. $\Psi_{X^0} \circ L_{X_\eta/\eta}^\pm(\mathcal{A}) = \mathbb{Z}/n\langle -1 \rangle \oplus \mathbb{Z}/n\langle 1 \rangle$. Hence, Theorem 3.3 implies on compact cohomology

$$R\Gamma_c(X_s^+, \Psi_X(\mathcal{A})) = \mathbb{Z}/n\langle -1 \rangle \oplus \mathbb{Z}/n\langle 1 \rangle.$$

i.e. $R\Gamma_c((\mathbb{A}_s^1)^+, \Psi_X(\mathcal{A}))$ is $\mathbb{Z}/n\langle -1 \rangle$ (resp. 0) on the flat (resp. non-flat) copy of \mathbb{A}^1 in (3.6).

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